MA119 Worksheet

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1 Integer Exponents

1.1 Properties of Exponents

The n-th power of a real number x, denoted as x^n, is defined as

$$x^n = \underbrace{x \cdot x \cdots x}_{n \text{ factors of } x}.$$

In the notation x^n, n is called ***the exponent***, x is called ***the base***, and x^n is called ***the power***.

- The **product rule** $x^m \cdot x^n = x^{m+n}$.
- The **quotient rule** (for $x \neq 0$.) $\frac{x^m}{x^n} = \begin{cases} x^{m-n} & \text{if} m \geq n. \\ \frac{1}{x^{n-m}} & \text{if} m \leq n. \end{cases}$
- The **zero-exponent rule** (for $x \neq 0$.) $x^0 = 1$.
- The **negative-exponent rule** (for $x \neq 0$.) $x^{-n} = \frac{1}{x^n}$ and $\frac{1}{x^{-n}} = x^n$.
- The **power-to-power** rule $(x^a)^b = x^{ab}$.
- The **product-to-power** rule $(xy)^n = x^n y^n$.
- The **quotient-to-power** rule (for $y \neq 0$.) $\left(\frac{x}{y}\right)^n = \frac{x^n}{y^n}$.

Example 1.1. Simplify. **Write with positive exponents.**

(1) $(-2x^2y^3z)(3xy^2z^2)$

(2) $\dfrac{2x^3y^2z}{6xy^3z^2}$

(3) $(-2^{10}x^3a^2p^0)^0$

(4) $(-3)^{-2}$

(5) $\left(\frac{(xy)^2}{xy^3}\right)^4$

(6) $\left(\frac{2y^{-2}z^{-5}}{4x^{-3}y^6}\right)^{-4}$.

1.2 Practice

Exercise 1.1. Simplify. **Write with positive exponents.**

(1) $(3a^2b^3c^2)(4abc^2)(2b^2c^3)$

(2) $\frac{4y^3z^0}{x^2y^2}$

(3) $(-2)^{-3}$

(4) $\frac{-u^0v^{15}}{v^{16}}$

(5) $(-2a^3b^2c^0)^3$

(6) $\frac{m^5n^2}{(mn)^3}$

(7) $(-3a^2x^3)^{-2}$

(8) $\left(\frac{-x^0y^3}{2wz^2}\right)^3$

(9) $\frac{3^{-2}a^{-3}b^5}{x^{-3}y^{-4}}$

(10) $\left(-x^{-1}(-y)^2\right)^3$

(11) $\left(\frac{6x^{-2}y^5}{2y^{-3}z^{-11}}\right)^{-3}$

(12) $\frac{(3x^2y^{-1})^{-3}(2x^{-3}y^2)^{-1}}{(x^6y^{-5})^{-2}}$

Exercise 1.2. A store has large size and small size watermelons. A large one cost \$4 and a small one \$1. Putting on the same table, a smaller watermelons has only half the height of the larger one. Given \$4, will you buy a large watermelon or 4 smaller ones? Why?

2 Review of Factoring

2.1 Factor by Removing the GCF

The greatest common factor (GCF) of two polynomials is a polynomial that all common factors divide it.

To ***factor a polynomial*** is to **express the polynomial as a product** of polynomials of lower degrees.

Example 2.1. Factor $4x^3y - 8x^2y^2 + 12x^3y^3$.

2.2 Factor by Grouping

Example 2.2. Factor $2x^2 - 6xy + xz - 3yz$.

Example 2.3. Factor $ax + 4b - 2a - 2bx$.

2.3 Factor Difference of Powers

$a^n - b^n = (a-b)(a^{n-1} + a^{n-2}b + \cdots + ab^{n-2} + b^{n-1})$

In particular,

$$a^2 - b^2 = (a-b)(a+b).$$

Example 2.4. Factor $25x^2 - 16$.

Example 2.5. Factor $32x^3y - 2xy^5$ completely.

2.4 Factor Trinomials

If a trinomial $ax^2 + bx + c$, $A \neq 0$, can be factored, then it can be expressed as a product of two binomials:

$$ax^2 + bx + c = (mx + n)(px + q),$$

where m, n, p and q satisfies the following equations.

$$a = \underbrace{mn}_{\text{F}} \qquad b = \underbrace{mq}_{\text{O}} \underset{+}{+} \underbrace{np}_{\text{I}} \qquad c = \underbrace{nq}_{\text{F}}$$

A trinomial $ax^2 + bx + c$ is also called a ***quadratic polynomial***.

Example 2.6. Factor $x^2 + 6x + 8$.

Example 2.7. Factor $2x^2 + 5x - 3$.

Example 2.8. Factor the trinomial completely.

$$4x^4 - x^2 - 3$$

2.5 Practice

Exercise 2.1. Factor out the GCF.

(1) $18x^2y^2 - 12xy^3 - 6x^3y^4$

(2) $5x(x-7) + 3y(x-7)$

(3) $-2a^2(x+y) + 3a(x+y)$

Exercise 2.2. Factor by grouping.

(1) $12xy - 10y + 18x - 15$

(2) $12ac - 18bc - 10ad + 15bd$

(3) $5ax - 4bx - 5ay + 4by$

Exercise 2.3. Factor completely.

(1) $25x^2 - 4$

(2) $8x^3 - 2x$

(3) $25xy^2 + x$

Exercise 2.4. Factor completely.

(1) $3x^3 + 6x^2 - 12x - 24$

(2) $x^4 + 3x^3 - 4x^2 - 12x$

Exercise 2.5. Factor the trinomial.

(1) $x^2 + 4x + 3$

(2) $x^2 + 6x - 7$

(3) $x^2 - 3x - 10$

Exercise 2.6. Factor the trinomial.

(1) $5x^2 + 7x + 2$

(2) $2x^2 + 5x - 12$

(3) $3x^2 - 10x - 8$

Exercise 2.7. Factor completely into polynomials with integer coefficients.

(1) $x^3 - 5x^2 + 6x$

(2) $4x^4 - 12x^2 + 5$

(3) $2x^3y - 9x^2y^2 - 5xy^3$

Exercise 2.8. Each of trinomial below has a factor in the table. Match the letter on the left of a factor with a number on the left a trinomial to decipher the following quotation.

$\overline{13} \quad \overline{10\ \ 2\ \ 9\ \ 15}, \quad \overline{9\ \ 5\ \ 14} \quad \overline{13} \quad \overline{4\ \ 3\ \ 15\ \ 7\ \ 2\ \ 1};$

$\overline{13} \quad \overline{11\ \ 2\ \ 2}, \quad \overline{9\ \ 5\ \ 14} \quad \overline{13} \quad \overline{8\ \ 5\ \ 3\ \ 6};$

$\overline{13} \quad \overline{14\ \ 3}, \quad \overline{9\ \ 5\ \ 14} \quad \overline{13} \quad \overline{12\ \ 5\ \ 14\ \ 2\ \ 15\ \ 11\ \ 1\ \ 9\ \ 5\ \ 14}.$

A: $3x-2$	**B:** $2x+1$	**C:** $x+6$	**D:** $x+7$	**E:** $2x-1$	**F:** $3x-1$	**G:** $x+10$
H: $x-8$	**I:** $2x+9$	**J:** $x-1$	**K:** $x+3$	**L:** $2x-5$	**M:** $x+5$	**N:** $x-7$
O: $x-13$	**P:** $5x-3$	**Q:** $4x-11$	**R:** $x-9$	**S:** $2x+3$	**T:** $x+4$	**U:** $7x+1$
V: $3x+5$	**W:** $3x+4$	**X:** $8x+3$	**Y:** $x-14$	**Z:** $5x-6$		

(1) $x^2-2x-24$
(2) $6x^2+x-2$
(3) $x^2-16x+39$
(4) $6x^2+13x-5$
(5) $x^2-5x-14$
(6) $3x^2-5x-12$
(7) $x^2-x-110$
(8) x^2-9
(9) $-3x^2+11x-6$
(10) $x^2-10x+16$
(11) $-2x^2+5x+12$
(12) $42x^2-x-1$
(13) $-2x^2-3x+27$
(14) $x^2+14x+49$
(15) x^2-81

3 Algebra of Rational Expressions

3.1 Simplified rational expressions

Let $p(x)$ and $q(x)$ be polynomials in on variable x and q is not a constant function. We call the quotient $r(x) = \frac{p(x)}{q(x)}$ a ***rational expression***. A rational expression is ***simplified*** if the numerator and the denominator have no common factor other than 1.

Two rational expressions are ***equivalent*** if they simplify into the same rational expression.

Let $p(x)$, $q(x)$ be polynomials with $q(x) \neq 0$ and $c(x)$ be a nonzero expression. Then

$$\frac{p(x) \cdot c(x)}{q(x) \cdot c(x)} = \frac{p(x)}{q(x)}.$$

Example 3.1. Simplify $\dfrac{x^2 + 4x + 3}{x^2 + 3x + 2}$.

Example 3.2. Simplify $\dfrac{2x^2 - x - 3}{2x^2 - 3x - 5}$.

3.2 Multiplying Rational Expressions

If $p(x), q(x), s(x), t(x)$ are polynomials with $q(x) \neq 0$ and $t(x) \neq 0$, then

$$\frac{p(x)}{q(x)} \cdot \frac{s(x)}{t(x)} = \frac{p(x)s(s)}{q(x)t(x)}.$$

Example 3.3. Multiply and then simplify.

$$\frac{3x^2}{x^2 + x - 6} \cdot \frac{x^2 - 4}{6x}.$$

Example 3.4. Multiply and then simplify.

$$\frac{3x^2 - 8x - 3}{x^2 + 8x + 16} \cdot \frac{x^2 - 16}{5x^2 - 14x - 3}.$$

3.3 Dividing Rational Expressions

If $p(x), q(x), s(x), t(x)$ are polynomials where $q(x) \neq 0, s(x) \neq 0$ and $t(x) \neq 0$, then

$$\frac{p(x)}{q(x)} \div \frac{s(x)}{t(x)} = \frac{p(x)}{q(x)} \cdot \frac{t(x)}{s(x)} = \frac{p(x)t(x)}{q(x)s(x)}.$$

Example 3.5. Divide and then simplify.

$$\frac{2x + 6}{x^2 - 6x - 7} \div \frac{6x - 2}{2x^2 - x - 3}.$$

3.4 Adding or Subtracting Rational Expressions with the Same Denominator

If $p(x)$, $q(x)$ and $r(x)$ are polynomials with $r(x) \neq 0$, then

$$\frac{p(x)}{r(x)} + \frac{q(x)}{r(x)} = \frac{p(x) + q(x)}{r(x)} \quad \text{and}$$

$$\frac{p(x)}{r(x)} - \frac{q(x)}{r(x)} = \frac{p(x) - q(x)}{r(x)}.$$

Example 3.6. Add and simplify

$$\frac{x^2+4}{x^2+3x+2}+\frac{5x}{x^2+3x+2}.$$

Example 3.7. Subtract and simplify $\dfrac{2x^2}{2x^2-x-3}-\dfrac{3x+5}{2x^2-x-3}$.

3.5 Adding or Subtracting Rational Expressions with Different Denominators

To add or subtract rational expressions with different denominators, we need to rewrite the rational expressions to equivalent rational expressions with the same denominator, say the LCD.

Example 3.8. Find the LCD of $\dfrac{3}{x^2-x-6}$ and $\dfrac{6}{x^2-4}$.

Example 3.9. Subtract and simplify

$$\frac{x-3}{x^2-2x-8}-\frac{1}{x^2-4}$$

3.6 Simplifying Complex Rational Expressions

A ***complex rational expression*** is a rational expression whose denominator or numerator contains a rational expression.

Example 3.10. Simplify

$$\frac{\frac{2x-1}{x^2-1}+\frac{x-1}{x+1}}{\frac{x+1}{x-1}-\frac{1}{x^2-1}}$$

3.7 Practice

Exercise 3.1. Simplify.

(1) $\dfrac{3x^2-x-4}{x+1}$

(2) $\dfrac{2x^2-x-3}{2x^2+3x+1}$

(3) $\dfrac{x^2-9}{3x^2-8x-3}$

Exercise 3.2. Multiply and simplify.

(1) $\dfrac{x+5}{x+4} \cdot \dfrac{x^2+3x-4}{x^2-25}$

(2) $\dfrac{3x^2-2x}{x+2} \cdot \dfrac{3x^2-4x-4}{9x^2-4}$

(3) $\dfrac{6y-2}{3-y} \cdot \dfrac{y^2-6y+9}{3y^2-y}$

Exercise 3.3. Divide and simplify.

(1) $\dfrac{9x^2-49}{6} \div \dfrac{3x^2-x-14}{2x+4}$

(2) $\dfrac{x^2+3x-10}{2x-2} \div \dfrac{x^2-5x+6}{x^2-4x+3}$

(3) $\dfrac{y-x}{xy} \div \dfrac{x^2-y^2}{y^2}$

Exercise 3.4. Simplify.

$$\frac{-x^2+11x-18}{x^2-4x+4} \div \frac{x^2-5x-36}{x^2-7x+12} \cdot \frac{2x^2+7x-4}{x^2+2x-15}$$

Exercise 3.5. Add/subtract and simplify.

(1) $\dfrac{x^2+2x-2}{x^2+2x-15} + \dfrac{5x+12}{x^2+2x-15}$

(2) $\dfrac{3x-10}{x^2-25} - \dfrac{2x-15}{x^2-25}$

(3) $\dfrac{4}{(x-3)(x+2)} + \dfrac{3x-2}{x^2-x-6}$

Exercise 3.6. Find the LCD of rational expressions.

(1) $\dfrac{2x}{2x^2 - 5x - 3}$ and $\dfrac{x-1}{x^2 - x - 6}$

(2) $\dfrac{9}{7x^2 - 28x}$ and $\dfrac{2}{x^2 - 8x + 16}$

Exercise 3.7. Add and simplify.

(1) $\dfrac{x}{x+1} + \dfrac{x-1}{x+2}$

(2) $\dfrac{x+2}{2x^2 - x - 3} + \dfrac{1}{x^2 + 3x + 2}$

(3) $\dfrac{4}{x-3} + \dfrac{3x-2}{x^2 - x - 6}$

Exercise 3.8. Subtract and simplify.

(1) $\dfrac{3x+5}{x^2-7x+12}-\dfrac{3}{x-3}$

(2) $\dfrac{y}{y^2-5y-6}-\dfrac{7}{y^2-4y-5}$

(3) $\dfrac{2x-3}{x^2+3x-10}-\dfrac{x+2}{x^2+2x-8}$

Exercise 3.9. Simplify.

$$\frac{x+11}{7x^2-2x-5}+\frac{x-2}{x-1}-\frac{x}{7x+5}$$

Exercise 3.10. Subtract and simplify.

$$\frac{x-1}{x^2-3x}+\frac{4}{x^2-2\,x-3}-\frac{1}{x\,(x+1)}$$

Exercise 3.11. Simplify.

(1) $\dfrac{1+\dfrac{2}{x}}{1-\dfrac{2}{x}}$

(2) $\dfrac{\dfrac{1}{x^2}-1}{\dfrac{1}{x^2}-\dfrac{1}{x}}$

Exercise 3.12. Simplify.

(1) $\dfrac{\dfrac{x^2-y^2}{y^2}}{\dfrac{1}{x}-\dfrac{1}{y}}$

(2) $\dfrac{\dfrac{2}{(x+1)^2}-\dfrac{1}{x+1}}{1-\dfrac{4}{(x+1)^2}}$

(3) $$\dfrac{\dfrac{5x}{x^2-x-6}}{\dfrac{2}{x+2}+\dfrac{3}{x-3}}$$

(4) $$\dfrac{\dfrac{x+1}{x-1}+\dfrac{x-1}{x+1}}{\dfrac{x+1}{x-1}-\dfrac{x-1}{x+1}}$$

Exercise 3.13. Tim and Jim refill their cars at the same gas station twice last month. Each time Tim got \$20 gas and Jim got 8 gallon. Suppose they refill their cars on same days. The price was \$2.5 per gallon the first time. The price on the second time changed. Can you find out who had the better average price?

4 Radicals and Rational Exponents

4.1 Radical Expressions

If $b^2 = a$, then we say that b is a ***square root*** of a. We denote the positive square root of a as $\sqrt{a}$, called the ***principal square root***.

For any real number a, the expression $\sqrt{a^2}$ can be simplified as

$$\sqrt{a^2} = |a|.$$

If $b^3 = a$, then we say that b is a ***cube root*** of a. The cube root of a real number a is denoted by $\sqrt[3]{a}$.

For any real number a, the expression $\sqrt[3]{a^3}$ can be simplified as

$$\sqrt[3]{a^3} = a.$$

In general, if $b^n = a$, then we say that b is an ***n-th root*** of a. If n is **even**, the **positive** n-th root of a, called the ***principal n-th root***, is denoted by $\sqrt[n]{a}$. If n is odd, the n-the root $\sqrt[n]{a}$ of a has the same sign with a.

In $\sqrt[n]{a}$, the symbol $\sqrt{\ }$ is called the ***radical sign***, a is called the ***radicand***, and n is called the ***index***.

If n is even, then the n-th root of a negative number is not a real number.

For any real number a, the expression $\sqrt[n]{a^n}$ can be simplified as

(1) $\sqrt[n]{a^n} = |a|$ if n is even.

(2) $\sqrt[n]{a^n} = a$ if n is odd.

A radical is simplified if the radicand has no perfect power factors against the radical.

Example 4.1. Simplify the radical expression using the definition.

(1) $\sqrt{4(y-1)^2}$

(2) $\sqrt[3]{-8x^3y^6}$

(3) $\sqrt{4(y-1)^2} = \sqrt{[2(y-1)]^2} = 2|y-1|$.

(4) $\sqrt[3]{-8x^3y^6} = \sqrt[3]{(-2xy^2)^3} = -2xy^2$.

4.2 Rational Exponents

If $\sqrt[n]{a}$ is a real number, then we define $a^{\frac{m}{n}}$ as
$$a^{\frac{m}{n}} = \sqrt[n]{a^m} = (\sqrt[n]{a})^m.$$

Rational exponents have the same properties as integral exponents:

(1) $a^m \cdot a^n = a^{m+n}$

(2) $\dfrac{a^m}{a^n} = a^{m-n}$

(3) $a^{-\frac{m}{n}} = \dfrac{1}{a^{\frac{m}{n}}}$

(4) $(a^m)^n = a^{mn}$

(5) $(ab)^m = a^m \cdot b^m$

(6) $\left(\dfrac{a}{b}\right)^m = \dfrac{a^m}{b^m}$

Example 4.2. Simplify the radical expression or the expression with rational exponents. **Write in radical notation**.

(1) $\sqrt{x}\sqrt[3]{x^2}$

(2) $\sqrt[3]{\sqrt{x^3}}$

(3) $\left(\dfrac{x^{\frac{1}{2}}}{x^{-\frac{5}{6}}}\right)^{\frac{1}{4}}$

(4) $\sqrt{\dfrac{x^{-\frac{1}{2}}y^2}{x^{\frac{3}{2}}}}$

In general, rewriting radical in rational exponents helps simplify calculations.

4.3 Product and Quotient Rules for Radicals

If $\sqrt[n]{a}$ and $\sqrt[n]{b}$ are real numbers, then

$$\sqrt[n]{a}\sqrt[n]{b} = \sqrt[n]{ab}.$$

If $\sqrt[n]{a}$ and $\sqrt[n]{b}$ are real numbers and $b \neq 0$, then

$$\frac{\sqrt[n]{a}}{\sqrt[n]{b}} = \sqrt[n]{\frac{a}{b}}.$$

Example 4.3. Simplify the expression.

(1) $\sqrt[4]{8xy^4}\,\sqrt[4]{2x^7y}$.

(2) $\dfrac{\sqrt[5]{96x^9y^3}}{\sqrt[5]{3x^{-1}y}}$.

4.4 Combining Like Radicals

Two radicals are called ***like radicals*** if they have the same index and the same radicand. We add or subtract like radicals by combining their coefficients.

Example 4.4. Simplify the expression.

$$\sqrt{8x^3} - \sqrt{(-2)^2x^4} + \sqrt{2x^5}.$$

4.5 Multiplying Radicals

Multiplying radical expressions with many terms is similar to that multiplying polynomials with many terms.

Example 4.5. Simplify the expression.

$$(\sqrt{2x} + 2\sqrt{x})(\sqrt{2x} - 3\sqrt{x}).$$

4.6 Rationalizing Denominators

Rationalizing denominator means rewriting a radical expression into an equivalent expression in which the denominator no longer contains radicals.

Example 4.6. Rationalize the denominator.

(1) $\dfrac{1}{2\sqrt{x^3}}$

(2) $\dfrac{\sqrt{x}+\sqrt{y}}{\sqrt{x}-\sqrt{y}}$

4.7 Complex Numbers

The imaginary unit i is defined as $\mathrm{i} = \sqrt{-1}$. Hence $\mathrm{i}^2 = -1$.

If b is a positive number, then $\sqrt{-b} = \mathrm{i}\sqrt{b}$.

Let a and b are two real numbers. We define a complex number by the expression $a + b\mathrm{i}$. The number $a $ is called the real part and the number b is called the imaginary part. If $b = 0$, then the complex number $a + b\mathrm{i} = a$ is just the real number. If $b \neq 0$, then we call the complex number $a+b\mathrm{i}$ an imaginary number. If $a = 0$ and $b \neq 0$, then the complex number $a + b\mathrm{i} = b\mathrm{i}$ is called a purely imaginary number.

Adding, subtracting, multiplying, dividing or simplifying complex numbers are similar to those for radical expressions. In particular, adding and subtracting become similar to combining like terms.

Example 4.7. Simplify and write your answer in the form $a+b\mathrm{i}$, where a and b are real numbers and i is the imaginary unit.

(1) $\sqrt{-3}\sqrt{-4}$

(2) $(4\mathrm{i} - 3)(-2 + \mathrm{i})$

(3) $\dfrac{-2 + 5\mathrm{i}}{\mathrm{i}}$

(4) $\dfrac{1}{1 - 2\mathrm{i}}$

(5) i^{2018}

Example 4.8. Evaluate the express $z^2 + \dfrac{z - 1}{z + 1}$ for $z = 1 + \mathrm{i}$. Write your answer in the form $a + b\mathrm{i}$.

4.8 Practice

Exercise 4.1. Evaluate the square root. If the square root is not a real number, state so.

(1) $-\sqrt{\dfrac{4}{25}}$

(2) $\sqrt{49} - \sqrt{9}$

Exercise 4.2. Simplify the radical expression.

(1) $\sqrt{(-7x^2)^2}$

(2) $\sqrt{(x+2)^2}$

Exercise 4.3. Simplify the radical expression.

(1) $\sqrt[3]{-27x^3}$

(2) $\sqrt[5]{(2x-1)^5}$

Exercise 4.4. Simplify the radical expression. Assume all variables are positive.

(1) $\sqrt{50}$

(2) $\sqrt[3]{-8x^2y^3}$

Exercise 4.5. Write the radical expression with rational exponents.

(1) $\sqrt[3]{(2x)^5}$

(2) $(\sqrt[5]{3xy})^7$

(3) $\sqrt[4]{(x^2+3)^3}$

Exercise 4.6. Write in radical notation and simplify.

(1) $4^{\frac{3}{2}}$

(2) $-81^{\frac{3}{4}}$

(3) $\left(\frac{27}{8}\right)^{-\frac{2}{3}}$

Exercise 4.7. Simplify the expression. Write with radical notations. Assume all variables represent nonnegative numbers.

(1) $\dfrac{12x^{\frac{1}{2}}}{4x^{\frac{2}{3}}}$

(2) $(x^{-\frac{3}{5}}y^{\frac{1}{2}})^{\frac{1}{3}}$

(3) $\left(\dfrac{x^{\frac{1}{2}}}{x^{-\frac{1}{3}}}\right)^4$

Exercise 4.8. Simplify the expression. Write in radical notation. Assume x is nonnegative.

(1) $\dfrac{\sqrt{x}}{\sqrt[3]{x}}$

(2) $\sqrt{\sqrt[3]{x}}$

(3) $\sqrt{x}\,\sqrt[3]{x}$

Exercise 4.9. Simplify the expression. Write in radical notation. Assume x is nonnegative.

(1) $\sqrt[5]{32x^{\frac{1}{3}}}$

(2) $\left(\dfrac{\sqrt[4]{9x}}{3}\right)^{-2}$

Exercise 4.10. Simplify the expression. Write in radical notation. Assume all variables are nonnegative.

(1) $\left(\dfrac{8a^{-\frac{5}{2}}b}{a^{\frac{1}{2}}b^{-5}}\right)^{-\frac{2}{3}}$

(2) $\left(\dfrac{y^{-\frac{1}{3}}}{\sqrt[3]{x^2}}\right)^{-3}$

(3) $\sqrt[3]{(-x)^{-2}}\sqrt{x^3}$

Exercise 4.11. Multiply and simplify.

(1) $\sqrt[3]{4}\sqrt[3]{5}$

(2) $\sqrt{|x+7|}\sqrt{|x-7|}$

(3) $\sqrt[3]{(x-y)^{\frac{5}{2}}}\sqrt[3]{(x-y)^{\frac{7}{2}}}$

Exercise 4.12. Simplify the radical expression. Assume all variables are positive.

(1) $\sqrt{20xy} \cdot \sqrt{4xy^2}$

(2) $\sqrt[5]{8x^4y^3z^3} \cdot \sqrt[5]{8xy^4z^8}$

Exercise 4.13. Divide. Assume all variables are positive. Answers must be simplified.

(1) $\sqrt{\dfrac{9x^3}{y^8}}$

(2) $\dfrac{\sqrt[3]{24a^6b^4}}{\sqrt[3]{3b}}$

Exercise 4.14. Add or subtract, and simplify. Assume all variables are positive.

(1) $4\sqrt{20} - 2\sqrt{5}$

(2) $3\sqrt{32x^2} + 5x\sqrt{8}$

Exercise 4.15. Add or subtract, and simplify. Assume all variables are positive

(1) $7\sqrt{4x^2} + 2\sqrt{25x} - \sqrt{16x}$

(2) $5\sqrt[3]{x^2y} + \sqrt[3]{27x^5y^4}$

(3) $3\sqrt{9y^3} - 3y\sqrt{16y} + \sqrt{25y^3}$

Exercise 4.16. Multiply and simplify. Assume all variables are positive.

(1) $\sqrt{2}(3\sqrt{3} - 2\sqrt{2})$

(2) $(\sqrt{5} + \sqrt{7})(3\sqrt{5} - 2\sqrt{7})$

(3) $(\sqrt{3} + \sqrt{2})^2$

(4) $(\sqrt{x+1} - 1)(\sqrt{x+1} + 1)$

(5) $(2\sqrt[3]{x}+6)(\sqrt[3]{x}+1)$

Exercise 4.17. Simplify the radical expression and rationalize the denominator. Assume all variables are positive.

(1) $\sqrt{\dfrac{2x}{7y}}$

(2) $\dfrac{\sqrt[3]{x}}{\sqrt[3]{3y^2}}$

Exercise 4.18. Simplify the radical expression and rationalize the denominator. Assume all variables are positive.

(1) $\dfrac{6\sqrt{3}}{\sqrt{3}-1}$

(2) $\dfrac{\sqrt{5}-\sqrt{3}}{\sqrt{5}+\sqrt{3}}$

(3) $\dfrac{3+\sqrt{2}}{2+\sqrt{3}}$

(4) $\dfrac{2\sqrt{x}}{\sqrt{x}-\sqrt{y}}$

Exercise 4.19. Simplify and rationalize the denominator. Assume all variables are positive.

(1) $\dfrac{\sqrt{x}}{\sqrt{x}-1}+\dfrac{1}{\sqrt{x}+1}$

(2) $\dfrac{\sqrt{x}+1}{\sqrt{x}}-\dfrac{1}{\sqrt{x}-1}$

Exercise 4.20. Add, subtract, multiply complex numbers and write your answer in the form $a+b\mathrm{i}$.

(1) $\sqrt{-2}\cdot\sqrt{-3}$

(2) $\sqrt{2} \cdot \sqrt{-8}$

(3) $(5 - 2\mathrm{i}) + (3 + 3\mathrm{i})$

(4) $(2 + 6\mathrm{i}) - (12 - 4\mathrm{i})$

Exercise 4.21. Add, subtract, multiply complex numbers and write your answer in the form $a + b\mathrm{i}$.

(1) $(3 + \mathrm{i})(4 + 5\mathrm{i})$

(2) $(7 - 2\mathrm{i})(-3 + 6\mathrm{i})$

(3) $(3 - x\sqrt{-1})(3 + x\sqrt{-1})$

(4) $(2+3\mathrm{i})^2$

Exercise 4.22. Divide the complex number and write your answer in the form $a+b\mathrm{i}$.

(1) $\dfrac{2\mathrm{i}}{1+\mathrm{i}}$

(2) $\dfrac{5-2\mathrm{i}}{3+2\mathrm{i}}$

(3) $\dfrac{2+3\mathrm{i}}{3-\mathrm{i}}$

(4) $\dfrac{4+7\mathrm{i}}{-3\mathrm{i}}$

Exercise 4.23. Simplify the expression.

(1) $(-\mathrm{i})^8$

(2) i^{2021}

(3) $\dfrac{1}{\mathrm{i}^{2022}}$

Exercise 4.24. Evaluate the function polynomial $2x^2 - 3x + 5$ for $x = 1 - \mathrm{i}$. Write your answer in the form $a + b\mathrm{i}$.

Exercise 4.25. Evaluate the polynomial $\mathrm{i}x^2 - x + \dfrac{2}{x-1}$ for $x = \mathrm{i} - 1$. Write your answer in the form $a + b\mathrm{i}$.

5 Solving by Factoring

5.1 Properties of Equations

Two equations are said to be ***equivalent*** if and only if they have the same solution set.

TRANSFORMS THAT CAN BE USED TO SOLVE AN EQUATION:

- **Adding or subtracting** the same quantity to both sides of an equation. For example, $x-1 = 2$ is equivalent to $x-1+1 = 2+1$.
- **Multiplying or dividing** both sides of an equation **by a nonzero quantity**. For example $2x = 4$ is equivalent to $\frac{2x}{2} = \frac{4}{2}$. Note that multiplying an expression to both sides may create solutions which are not solutions of the original equation. Such a solution is called an ***extraneous solutions***.
- **Applying an identity** to transform one side of the equation. For example, $x^2 - 1 = 0$ is equivalent to $(x-1)(x+1) = 0$, where the identity $x^2 - 1 = (x-1)(x+1)$ was applied.
- **Applying a function** to both sides of the equation. For example, taking square of both sides of the equation $\sqrt{x} = 2$ yields a new equation $x = 4$. In general, solutions of the resulting equation include solution(s) of the original equation and may also have extraneous solutions. For example, taking squares of both sides of the equation $x = 1$ produces the equation $x^2 = 1$. The new equation $x^2 = 1$ has two solutions $x = -1$ and $x = 1$, but the original equation $x = 1$ only has one solution. The solution $x = -1$ of the equation $x^2 = 1$ is an extraneous solution of the equation $x = 1$.

5.2 Quadratic Equations

A **polynomial equation** is an equation that can be written in the form

$$a_n x^n + a_{n-1}x^{n-1} + \cdots + a_2x^2 + a_1x + a_0 = 0,$$

where n is a positive integer and $a_n \neq 0$.

A polynomial equation is called a ***quadratic equation*** if $n = 2$. For example, $2x^2 + 5x - 3 = 0$. We often write a quadratic equation in its ***standard form***

$$ax^2 + bx + c = 0,$$

where a, b and c are numbers, and $a \neq 0$.

The ***Zero-product property***:

$$A \cdot B = 0 \quad \text{if and only if} \quad A = 0 \text{ or } B = 0,$$

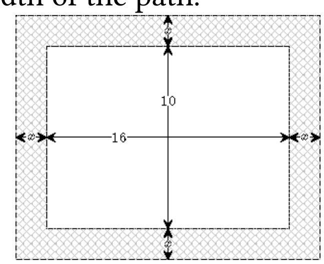
Example 5.1. Solve the equation

$$2x^2 + 5x = 3.$$

Example 5.2. Solve the equation

$$(x - 2)(x + 3) = -4.$$

Example 5.3. A rectangular garden is surrounded by a path of uniform width. If the dimension of the garden is 10 meters by 16 meters and the total area is 216 square meters, determine the width of the path.



5.3 Practice

Exercise 5.1. Solve the equation by factoring.

(1) $x^2 - 3x + 2 = 0$

(2) $2x^2 - 3x = 5$

(3) $(x-1)(x+3) = 5$

(4) $\frac{1}{3}(2-x)(x+5) = 4$

Exercise 5.2. Find all real solutions of the equation by factoring.

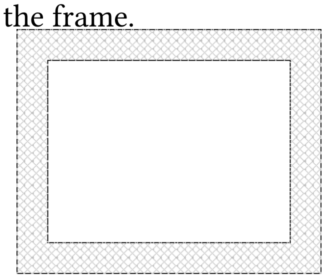
(1) $4(x-2)^2 - 9 = 0$

(2) $2x^3 - 18x = 0$

(3) $3x^4 - 2x^2 = 1$

(4) $x^3 - 3x^2 - 4x + 12 = 0$

Exercise 5.3. A paint measuring 3 inches by 4 inches is surrounded by a frame of uniform width. If the combined area of the paint and the frame is 30 square inches, determine the width of the frame.



Exercise 5.4. A rectangle whose length is 2 meters longer than its width has an area 8 square meters. Find the width and the length of the rectangle.

Exercise 5.5. The product of two **consecutive negative odd** numbers is 35. Find the numbers.

Exercise 5.6. In a right triangle, the long leg is 2 inches more than double of the short leg. The hypotenuse of the triangle is 1 inch longer than the long leg. Find the length of the shortest side.

Exercise 5.7. A ball is thrown upwards from a rooftop. It will reach a maximum vertical height and then fall back to the ground. The height $h(t)$ of the ball from the ground after time t seconds is $h(t) = -16t^2 + 48t + 160$ feet. How long it will take the ball to hit the ground?

Exercise 5.8. A toy factory estimates that the demand of a particular toy is $300 - x$ units each week if the price is $\$x$ dollars per unit.

(1) Find the function that models the weekly revenue, R, received when the selling price is $\$x$ per unit.

(2) What the price range so the revenue is nonnegative.

6 Completing the Square

The square root property:
Suppose that $X^2 = d$. Then $X = \sqrt{d}$ or $X = -\sqrt{d}$, or simply $X = \pm\sqrt{d}$.

The square root property provides another method to solve a quadratic equation, completing the square.

Let $h = -\frac{b}{2a}$ and $k = ah^2 + bh + c$. Then

$$ax^2 + bx + c = a(x-h)^2 + k = a\left(x + \frac{b}{2a}\right)^2 + \frac{4ac - b^2}{4a^2}.$$

The procedure to rewrite a trinomial as the sum of a perfect square and a constant is called ***completing the square***.

Example 6.1. Solve the equation $x^2 + 2x - 1 = 0$.

Example 6.2. Solve the equation $-2x^2 + 8x - 9 = 0$.

6.1 The Quadratic Formula

Using the method of completing the square, we obtain the follow quadratic formula for the quadratic equation $ax^2+bx+c = 0$ with $a \neq 0$:

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}.$$

The quantity b^2-4ac is called the ***discriminant*** of the quadratic equation.

(1) If $b^2 - 4ac > 0$, the equation has two real solutions.

(2) If $b^2 - 4ac = 0$, the equation has one real solution.

(3) If $b^2 - 4ac < 0$, the equation has two imaginary solutions (no real solutions).

Example 6.3. Determine the type and the number of solutions of the equation $(x-1)(x+2) = -3$.

Example 6.4. Solve the equation $2x^2 - 4x + 7 = 0$.

Example 6.5. Find the base and the height of a **triangle** whose base is three inches more than twice its height and whose area is 5 square inches. Round your answer to the nearest tenth of an inch.

6.2 Practice

Exercise 6.1. Solve the quadratic equation by the square root property.

(1) $2x^2 - 6 = 0$

(2) $(x-3)^2 = 10$

(3) $4(x+1)^2 + 25 = 0$

Exercise 6.2. Solve the quadratic equation by completing the square.

(1) $x^2 + x - 1 = 0$

(2) $x^2 + 8x + 12 = 0$

(3) $3x^2 + 6x - 1 = 0$

Exercise 6.3. Determine the number and the type of solutions of the given equation.

(1) $x^2 + 8x + 3 = 0$

(2) $3x^2 - 2x + 4 = 0$

(3) $2x^2 - 4x + 2 = 0$

Exercise 6.4. Solve using the quadratic formula.

(1) $x^2 + 3x - 7 = 0$

(2) $2x^2 = -4x + 5$

(3) $2x^2 = x - 3$

Exercise 6.5. Solve using the quadratic formula.

(1) $(x - 1)(x + 2) = 3$

(2) $2x^2 - x = (x+2)(x-2)$

(3) $\frac{1}{2}x^2 + x = \frac{1}{3}$

Exercise 6.6. A **triangle** whose area is 7.5 square meters has a base that is one meter less than triple the height. Find the length of its base and height. Round to the nearest hundredth of a meter.

Exercise 6.7. A **rectangular** garden whose length is 2 feet longer than its width has an area 66 square feet. Find the dimensions of the garden, rounded to the nearest hundredth of a foot.

7 Rational Equations

7.1 Solving Rational Equations

A ***rational equation*** is an equation that contains a rational expression. One way to solve rational equations is to clear all denominators by multiplying the LCD to both sides.

Example 7.1. Solve
$$\frac{5}{x^2-9}=\frac{3}{x-3}-\frac{2}{x+3}.$$

Another way to solve a rational equation is to rewrite and simplify the equation into the form $\frac{A}{B}=0$ where $\frac{A}{B}$ is a **reduced fraction**. Then the rational equation is equivalent to the equation $A=0$.

Example 7.2. Solve the quation
$$\frac{2x-3}{x-3}-\frac{1}{x+1}=\frac{4}{x^2-2x-3}$$

7.2 Literal Equations

A ***literal equation*** is an equation involving two or more variables. When solving a literal equation for one variable, other variables can be viewed as constants.

Example 7.3. Solve for x from the equation
$$\frac{1}{x}+\frac{1}{y}=\frac{1}{z}.$$

7.3 Practice

Exercise 7.1. Solve.

(1) $\dfrac{1}{x+1} + \dfrac{1}{x-1} = \dfrac{4}{x^2-1}$

(2) $\dfrac{30}{x^2-25} = \dfrac{3}{x+5} + \dfrac{2}{x-5}$

Exercise 7.2. Solve.

(1) $\dfrac{2x-1}{x^2+2x-8} = \dfrac{1}{x-2} - \dfrac{2}{x+4}$

(2) $\dfrac{3x}{x-5} = \dfrac{2x}{x+1} - \dfrac{42}{x^2-4x-5}$

Exercise 7.3. Solve a variable from a formula.

(1) Solve for f from $\dfrac{1}{p} + \dfrac{1}{q} = \dfrac{1}{f}$.

(2) Solve for x from $A = \dfrac{f + cx}{x}$.

Exercise 7.4. Solve for x from the equation.

(1) $2(x+1)^{-1} + x^{-1} = 2$.

(2) $\dfrac{a^2x + 2a}{x^{-1}} = -1$.

Exercise 7.5. David can row 3 miles per hour in still water. It takes him 90 minutes to row 2 miles upstream and then back. How fast is the current?

Exercise 7.6. The size of a A0 paper is defined to have an area of 1 square meter with the longer dimension $\sqrt[4]{2}$ meters. Successive paper sizes in the series A1, A2, A3, and so forth, are defined by halving the preceding paper size across the larger dimension. Can you find the dimension of a A4 paper?

8 Radical Equations

8.1 Solving Radical Equations by Taking a Power

The idea to solve a radical equation $\sqrt[n]{X} = a$ is to first take n-th power of both sides to get rid of the radical sign, that is $X = a^n$ and then solve the resulting equation.

Example 8.1. Solve the equation $x - \sqrt{x+1} = 1$.

Example 8.2. Solve the equation $\sqrt{x-1} - \sqrt{x-6} = 1$.

Example 8.3. Solve the equation $-2\sqrt[3]{x-4} = 6$.

8.2 Equations Involving Rational Exponents

Equation involving rational exponents may be solved similarly. However, one should be careful with meaning of the expression $\left(X^{\frac{m}{n}}\right)^{\frac{n}{m}}$. When m is even and n is odd, $\left(X^{\frac{m}{n}}\right)^{\frac{n}{m}} = |X|$. Otherwise, $\left(X^{\frac{m}{n}}\right)^{\frac{n}{m}} = X$.

Example 8.4. Solve the equation $(x+2)^{\frac{1}{2}} - (x-3)^{\frac{1}{2}} = 1$.

Example 8.5. Solve the equation $(x-1)^{\frac{2}{3}} = 4$.

8.3 Learn from Mistakes

Example 8.6. Can you find the mistakes made in the solution and fix it?

Solve the radical equation.

$$\sqrt{x-1} + 2 = x$$

Solution (incorrect):

$$\begin{aligned}
\sqrt{x-2} + 2 &= x \\
(\sqrt{x-2})^2 + 2^2 &= x^2 \\
x - 2 + 4 &= x^2 \\
x + 2 &= x^2 \\
x^2 - x - 2 &= 0 \\
(x-2)(x+1) &= 0
\end{aligned}$$

$$x - 2 = 0 \qquad \text{or} \qquad x + 1 = 0$$

$$x = 2 \qquad \text{or} \qquad x = -1$$

Answer: the equation has two solutions $x = 2$ and $x = -1$.

8.4 Practice

Exercise 8.1. Solve each radical equation.

(1) $\sqrt{3x+1} = 4$

(2) $\sqrt{2x-1} - 5 = 0$

Exercise 8.2. Solve each radical equation.

(1) $\sqrt{5x+1} = x + 1$

(2) $x = \sqrt{3x+7} - 3$

Exercise 8.3. Solve each radical equation.

(1) $\sqrt{6x+7} - x = 2$

(2) $\sqrt{x+2}+\sqrt{x-1}=3$

Exercise 8.4. Solve each radical equation.

(1) $\sqrt{x+5}-\sqrt{x-3}=2$

(2) $3\sqrt[3]{3x-1}=6$

Exercise 8.5. Solve each radical equation.

(1) $(x+3)^{\frac{1}{2}}=x+1$

(2) $2(x-1)^{\frac{1}{2}}-(x-1)^{-\frac{1}{2}}=1$

Exercise 8.6. Solve each radical equation.

(1) $(x-1)^{\frac{3}{2}}=8$

(2) $(x+1)^{\frac{2}{3}} = 4$

9 Absolute Value Equations

9.1 Properties of Absolute Values

The ***absolute value*** of a real number a, denoted by $|a|$, is the distance from.

Absolute values satisfy the following properties:

$$|-a| = |a|, \quad |ab| = |a||b| \quad \text{and} \quad \left|\frac{a}{b}\right| = \frac{|a|}{|b|}.$$

An **absolute value equation** may be rewritten as $|X| = c$, where X represents an algebraic expression.

If c is **positive**, then the equation $|X| = c$ is equivalent to $X = c \quad \text{or} \quad X = -c$.

If c is **negative**, then the solution set of $|X| = c$ is **empty**. An ***empty set*** is denoted by $\emptyset$.

More generally, $|X| = |Y|$ is equivalent to $X = Y$ or $X = -Y$.

The equation $|X| = 0$ is equivalent to $X = 0$.

Example 9.1. Solve the equation

$$|2x - 3| = 7.$$

Example 9.2. Solve the equation

$$|2x - 1| - 3 = 8.$$

Example 9.3. Solve the equation

$$3|2x - 5| = 9.$$

Example 9.4. Solve the equation

$$2|1 - 2x| - 3 = 7.$$

Example 9.5. Solve the equation

$$|3x - 2| = |x + 2|.$$

Example 9.6. Solve the equation

$$2|1 - x| = |2x + 10|.$$

9.2 Practice

Exercise 9.1. Find the solution set for the equation.

(1) $|2x - 1| = 5$

(2) $\left|\dfrac{3x-9}{2}\right| = 3$

Exercise 9.2. Find the solution set for the equation.

(1) $|3x-6|+4=13$

(2) $3|2x-5|=9$

Exercise 9.3. Find the solution set for the equation.

(1) $|5x-10|+6=6$

(2) $-3|3x-11|=5$

Exercise 9.4. Find the solution set for the equation.

(1) $3|5x-2|-4=8$

(2) $-2|3x+1|+5=-3$

Exercise 9.5. Find the solution set for the equation.

(1) $|5x-12|=|3x-4|$

(2) $|x-1|=-5|(2-x)-1|$

Exercise 9.6. Find the solution set for the equation.

(1) $|2x-1|=5-x$

(2) $-2x=|x+3|$

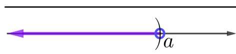
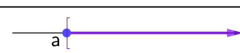
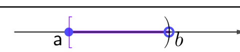
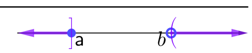
10 Linear Inequalities

10.1 Properties and Definitions

Property	Example
The additive property If $a < b$, then $a + c < b + c$, for any real number c. If $a < b$, then $a - c < b - c$, for any real number c.	If $x + 3 < 5$, then $x + 3 - 3 < 5 - 3$. Simplifying both sides, we get $x < 2$.
The positive multiplication property If $a < b$ and c is positive, then $ac < bc$. If $a < b$ and c is positive, then $\frac{a}{c} < \frac{b}{c}$.	If $2x < 4$, then $\frac{2x}{2} < \frac{4}{2}$. Simplifying both sides, we get $x < 2$.
The negative multiplication property If $a < b$ and c is negative, then $ac > bc$. If $a < b$ and c is negative, then $\frac{a}{c} > \frac{b}{c}$.	If $1 < 2$, then $-2 = 1 \cdot (-2) > 2 \cdot (-2) = -4$. If $-2x < 4$, then $\frac{-2x}{-2} > \frac{4}{-2}$. Simplifying both sides, we get $x > 2$.

A ***compound inequality*** is formed by two inequalities with the word *and* or the word *or*.

Solutions to an inequality normally form an interval which has boundaries and should reflect inequality signs. Depending on the form of an inequality, we may a single interval and a union of intervals. For example, suppose $a < b$, we have the following equivalent representations of inequalities.

$x < a$	$x \geq b$	$a \leq x < b$	$x \leq a$ or $x > b$
			
$(-\infty, a)$	$[b, \infty)$	$[a, b)$	$(-\infty, a] \cup (b, \infty)$

Example 10.1. Solve the linear inequality

$$2x + 4 > 0.$$

Example 10.2. Solve the linear inequality
$$-3x - 4 < 2.$$

Example 10.3. Solve the compound linear inequality
$$x + 2 < 3 \quad \text{and} \quad -2x - 3 < 1.$$

Example 10.4. Solve the compound linear inequality
$$-x + 4 > 2 \quad \text{or} \quad 2x - 5 \geq -3.$$

Example 10.5. Solve the compound linear inequality
$$-4 \leq \frac{2x - 4}{3} < 2.$$

Example 10.6. Solve the compound linear inequality
$$-1 \leq \frac{-3x+4}{2} < 3.$$

Example 10.7. Suppose that $-1 \leq x < 2$. Find the range of $5-3x$. Write your answer in interval notation.

10.2 Practice

Exercise 10.1. Solve the linear inequality. **Write your answer in interval notation.**

(1) $3x+7 \leq 1$

(2) $2x - 3 > 1$

Exercise 10.2. Solve the linear inequality. **Write your answer in interval notation.**

(1) $4x + 7 > 2x - 3$

(2) $3 - 2x \leq x - 6$

Exercise 10.3. Solve the compound linear inequality. **Write your answer in interval notation.**

(1) $3x + 2 > -1$ and $2x - 7 \leq 1$

(2) $4x - 7 < 5$ and $5x - 2 \geq 3$

Exercise 10.4. Solve the compound linear inequality. **Write your answer in interval notation.**

(1) $-4 \le 3x + 5 < 11$

(2) $7 \ge 2x - 3 \ge -7$

Exercise 10.5. Solve the compound linear inequality. **Write your answer in interval notation.**

(1) $3x - 5 > -2$ or $10 - 2x \le 4$

(2) $2x + 7 < 5$ or $3x - 8 \ge x - 2$

Exercise 10.6. Solve the compound linear inequality. **Write your answer in interval notation.**

(1) $-2 \le \dfrac{2x-5}{3} < 3$

(2) $-1 < \dfrac{3x+7}{2} \leq 4$

Exercise 10.7. Solve the linear inequality. **Write your answer in interval notation.**

$$\frac{1}{3}x + 1 < \frac{1}{2}(2x - 3) - 1$$

Exercise 10.8. Solve the compound linear inequality. **Write your answer in interval notation.**

$$0 \leq \frac{2}{5} - \frac{x+1}{3} < 1$$

Exercise 10.9. Suppose $0 < x \leq 1$. Find the range of $-2x + 1$. **Write your answer in interval notation.**

Exercise 10.10. Suppose that $x + 2y = 1$ and $1 \leq x < 3$. Find the range of y. **Write your answer in interval notation.**

Exercise 10.11. A toy store has a promotion "Buy one get the second one half price" on a certain popular toy. The sale price of the toy is \$20 each. Suppose the store makes more profit when you buy two. What do you think the store's purchasing price of the toy is?

11 Introduction to Functions

11.1 Definition and Notations

A ***relation*** is a set of ordered pairs. The set of all first components of the ordered pairs is called the ***domain***. The set of all second components of the ordered pairs is called the ***range***.

A ***function*** is a relation such that each element in the domain corresponds to **exactly one** element in the range.

For a function, we usually use the variable x to represent an element from the domain and call it the ***independent variable***. The variable y is used to represent the value corresponding to x and is called the ***dependent variable***. We say y is a function of x. When we consider several functions together, to distinguish them we named functions by a letter such as f, g, or F. The notation $f(x)$, read as **"f of x" or "f at x"**, represents the output of the function f when the input is x.

The ***domain*** of a function is the set of all allowed inputs. The ***range*** of a function is the set of all outputs.

Example 11.1. Find the indicated function value.

(1) $f(-2), \quad f(x) = 2x + 1$

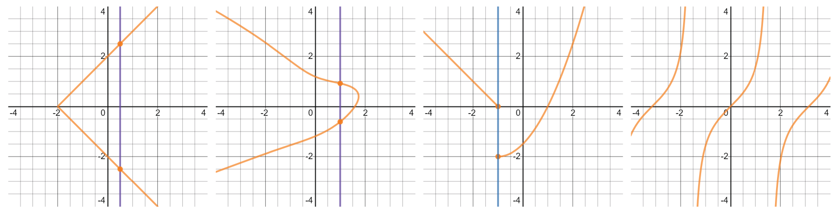
(2) $g(2), \quad g(x) = 3x^2 - 10$

(3) $h(a - t), \quad h(x) = 3x + 5.$

11.2 Graphs of Functions

The graph of a function is the graph of its ordered pairs. A graph of ordered pairs (x, y) in the rectangular coordinate system defines y as a function of x if any vertical line crosses the graph at most once. This test is called the ***vertical line test***.

Example 11.2. Determine which of the following graphs defines a function.



A B C D

11.3 Graph Reading

Example 11.3. Use the graph in the picture to answer the following questions.

(1) Determine whether the graph is a function and explain your answer.

(2) Find the domain (in interval notation) of the graph.

(3) Find the range (in interval notation) of the graph.

(4) Find the interval where the graph is above 2.

(5) Find the interval where the graph is is decreasing.

(6) Find all maximum and minimum values of the function if they exist.

(7) Find the value of y such that the point $(3, y)$ is on the graph.

(8) Find the value of x such that $(x, 0)$ is on the graph.

11.4 Practice

Exercise 11.1. Find the indicated function values for the functions $f(x) = -x^2 + x - 1$ and $g(x) = 2x - 1$. Simplify your answer.

(1) $f(2)$

(2) $f(-x)$

(3) $g(-1)$

(4) $g(f(1))$

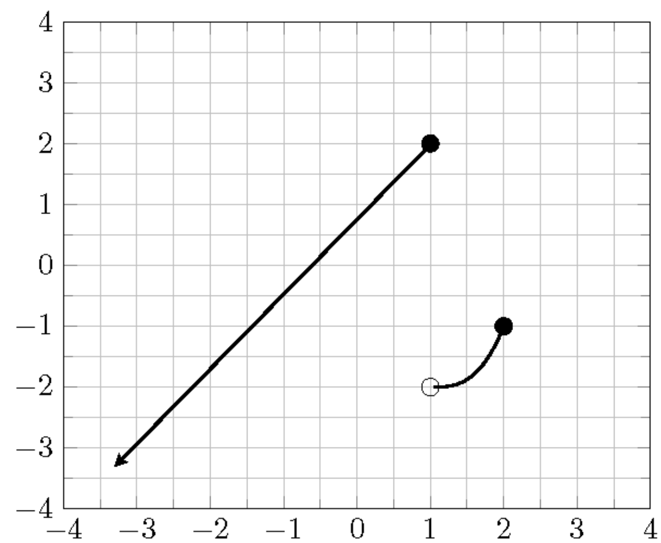
Exercise 11.2. Suppose $g(x) = -3x + 1$.

(1) Compute $\dfrac{g(4) - g(1)}{4 - 1}$

(2) Compute $\dfrac{g(x+h)-g(x)}{h}$

Exercise 11.3. Suppose the domain of the linear function $l(x) = 1 - 2x$ is $(0, 1)$. Find the range of the function.

Exercise 11.4. Use the graph in the picture to answer the following questions.



(1) Determine whether the graph is a function and explain your answer.

(2) Find the domain of the graph (write the domain in interval notation).

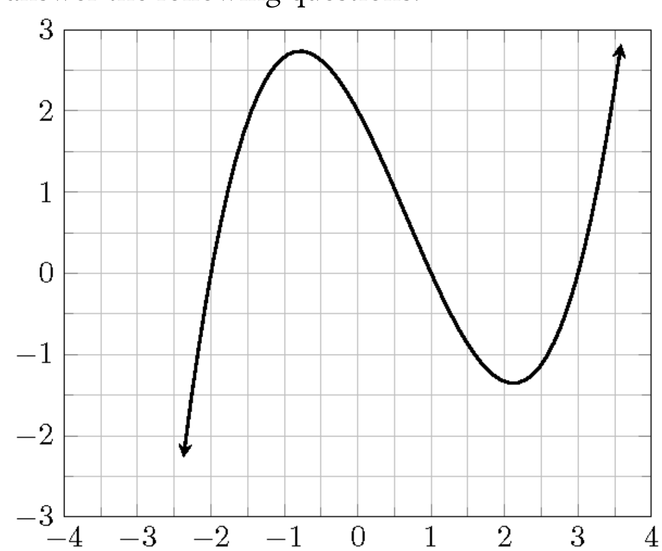
(3) Find the range of the graph (write the range in interval notation).

(4) Find the interval where the graph is above the x-axis.

(5) Find all points where the graph reaches a maximum or a minimum.

(6) Find the values of the x-coordinate of all points on the graph whose y-coordinate is 1.

Exercise 11.5. Use the graph of the function f in the picture to answer the following questions.



(1) Find the y-intercept.

(2) Find the value $\dfrac{f(3) - f(0)}{3}$.

(3) Find the values x such that $f(x) = 0$.

(4) Find the solution to the inequality $f(x) > 0$. Write in interval notation.

Exercise 11.6. Today Matt drove from home to school in 30 minutes. He spent 6 minutes on local streets before driving on the highway and 4 minutes on local streets towards school after getting off the highway. On local streets, his average speed is 30 miles per hour. On the highway, his average speed is 60 miles per hours.

(1) Write the distance d (in miles) he drove as a function of the time t (in minutes)?

(2) After 15 minutes, where was he and how far did he drive?

(3) How far did he drive from home to school?

12 Linear Functions

12.1 Cost, Revenue and Profit

A company has fixed costs of $10,000 for equipment and variable costs of $15 for each unit of output. The sale price for each unit is $25. What is total cost, total revenue and total profit at varying levels of output?

12.2 The Slope-Intercept Form Equation

The slope of a line measures the steepness, in other words, "rise" over "run", or rate of change of the line. Using the rectangular coordinate system, the ***slope*** m of a line is defined as

$$m = \frac{y_2 - y_1}{x_2 - x_1} = \frac{\text{rise}}{\text{run}} = \frac{\text{change in the output } y}{\text{change in the input } x},$$

where (x_1, y_1) and (x_2, y_2) are any two distinct points on the line. If the line intersects the y-axis at the point $(0, b)$, then a point (x, y) is on the line if and only if

$$y = mx + b.$$

This equation is called the ***slope-intercept form*** of the line.

12.3 Point-Slope Form Equation of a Line

Suppose a line passing through the point (x_0, y_0) has the slope m. Solving from the slope formula, we see that any point (x, y) on the line satisfies the equation equation

$$y = m(x - x_0) + y_0$$

which is called the ***point-slope form*** equation.

12.4 Linear Function

A ***linear function*** f is a function whose graph is a line. An equation for f can be written as

$$f(x) = mx + b$$

where m is the slope and $b = f(0)$.

A function f is a linear function if the following equalities hold
$$\frac{f(x_2)-f(x_1)}{x_2-x_1}=\frac{f(x_3)-f(x_1)}{x_3-x_1}$$
for any three distinct points (x_1, y_1), (x_2, y_2) and (x_3, y_3) on the graph of f.

12.5 Equations of Linear Functions

Example 12.1. Find the slope-intercept form equation for the linear function f such that $f(2) = 5$ and $f(-1) = 2$.

12.6 Graph a Linear Function by Plotting Points

Example 12.2. Sketch the graph of the linear function $f(x) = -\frac{1}{2}x + 1$.

12.7 Horizontal and Vertical Lines

A ***horizontal line*** is defined by an equation $y = b$. The slope of a horizontal line is simply zero. A ***vertical line*** is defined by an equation $x = a$. The slope of a vertical line is **undefined**.

A vertical line gives an example that a graph is not a function of x. Indeed, the vertical line test fails for a vertical line.

12.8 Explicit Function

When studying functions, we prefer a clearly expressed function rule. For example, in $f(x) = -\frac{2}{3}x + 1$, the expression $-\frac{2}{3}x + 1$ clearly tells us how to produce outputs. For a function f defined by an equation, for instance, $2x + 3y = 3$, to find the function rule (that is an expression), we simply solve the given equation for y.

$$\begin{aligned} 2x + 3y &= 3 \\ 3y &= -2x + 3 \\ y &= -\frac{2}{3}x + 1. \end{aligned}$$

Now, we get $f(x) = -\frac{2}{3}x + 1$.

12.9 Perpendicular and Parallel Lines

Any two vertical lines are parallel. Two non-vertical lines are ***parallel*** if and only if they **have the same slope**.

A line that is parallel to the line $y = mx + a$ has an equation $y = mx + b$, where $a \neq b$.

Horizontal lines are perpendicular to vertical lines. Two non-vertical lines are ***perpendicular*** if and only if **the product of their slopes is** -1.

A line that is perpendicular to the line $y = mx + a$ has an equation $y = -\frac{1}{m}x + b$.

12.10 Finding Equations for Perpendicular or Parallel Lines

Example 12.3. Find an equation of the line that is parallel to the line $4x + 2y = 1$ and passes through the point $(-3, 1)$.

Example 12.4. Find an equation of the line that is perpendicular to the line $4x - 2y = 1$ and passes through the point $(-2, 3)$.

12.11 Practice

Exercise 12.1. Find the slope of the line passing through

(1) $(3, 5)$ and $(-1, 1)$

(2) $(-2, 4)$ and $(5, -2)$.

Exercise 12.2. Find the point-slope form equation of the line with slope 5 that passes though $(-2, 1)$.

Exercise 12.3. Find the point-slope form equation of the line passing thought $(3, -2)$ and $(1, 4)$.

Exercise 12.4. Find the slope-intercept form equation of the line passing through $(6, 3)$ and $(2, 5)$.

Exercise 12.5. Determine whether the linear functions $f(x)$ and $h(x)$ with the following values $f(-2) = -4$ $f(0) = h(0) = 2$ and $h(2) = 8$ define the same function. Explain your answer.

Exercise 12.6. Suppose the points $(5, -1)$ and $(2, 5)$ are on the graph of a linear function f. Find $f(-3)$.

Exercise 12.7. Graph the functions.

(1) $f(x) = -x + 1$

(2) $f(x) = \frac{1}{2}x - 1$

Exercise 12.8. A storage rental company charges a base fee of \$15 and \$$x$ per day for a small cube. Suppose the cost is \$20 dollars for 10 days.

(1) Write the cost y (in dollars) as a linear function of the number of days x.

(2) How much would it cost to rent a small cube for a whole summer (June, July and August)?

Exercise 12.9. Find an equation for each of the following two lines which pass through the same point $(-1, 2)$.

(1) The vertical line.

(2) The horizontal line.

Exercise 12.10. Line L is defined by the equation $2x - 5y = -3$. What is the slope $m_{\parallel}$ of the line that is parallel to the line L? What is the slope $m_{\perp}$ of the line that is perpendicular to the line L.

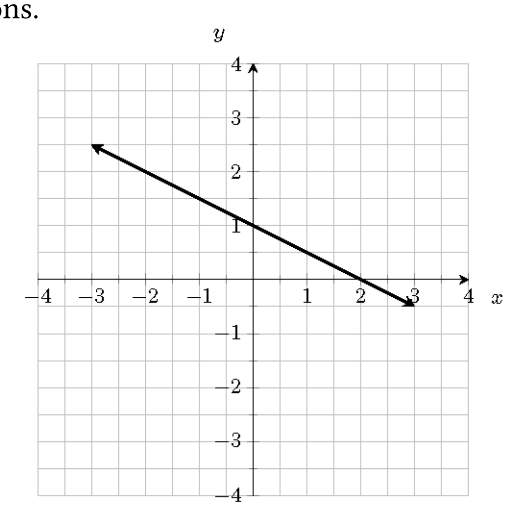
Exercise 12.11. Line L_1 is defined by $3y + 5x = 7$. Line L_2 passes through $(-1, -3)$ and $(4, -8)$. Determine whether L_1 and L_2 are parallel, perpendicular or neither.

Exercise 12.12. Find the point-slope form and then the slope-intercept form equations of the line parallel to $3x - y = 4$ and passing through the point $(2, -3)$.

Exercise 12.13. Find the slope-intercept form equation of the line that is perpendicular to $4y - 2x + 3 = 0$ and passing through the point $(2, -5)$

Exercise 12.14. The line L_1 is defined $Ax + By = 3$. The line L_2 is defined by the equation $Ax + By = 2$. The line L_3 is defined by $Bx - Ay = 1$. Determine whether L_1, L_2 and L_3 are parallel or perpendicular to each other.

Exercise 12.15. Use the graph of the line L to answer the questions.



(1) Find an equation for the line L.

(2) Find an equation for the line $L_{\perp}$ perpendicular to L and passing through $(1, 1)$.

(3) Find an equation for the line $L_{\parallel}$ parallel to L and passing through $(-2, -1)$.

Exercise 12.16. Determine whether the points $(-3, 1)$, $(-2, 6)$, $(3, 5)$ and $(2, 0)$ form a square. Please explain your conclusion.

13 Quadratic Functions

13.1 Maximize the Revenue

When price increases, demand decreases and vice verse. A retail store found that the price p as a function of the demand x for a certain product is $p(x) = 100 - \frac{1}{2}x$. The revenue R of selling x units is $R = x \cdot p(x) = x(100 - \frac{1}{2}x)$. To maximize the revenue, what should be the price?

13.2 The Graph of a Quadratic Function

The graph of a quadratic function $f(x) = ax^2 + bx + c$, $a \neq 0$, is called a ***parabola***.

By completing the square, a quadratic function $f(x) = ax^2 + bx + c$ can always be written in the form $f(x) = a(x-h)^2 + k$, where $h = -\dfrac{b}{2a}$ and $k = f(h) = f\left(-\dfrac{b}{2a}\right)$.

(1) The line $x = h = -\dfrac{b}{2a}$ is called the ***axis of symmetry*** of the parabola.

(2) The point $(h, k) = \left(-\dfrac{b}{2a}, f\left(-\dfrac{b}{2a}\right)\right)$ is called the ***vertex*** of the parabola.

13.3 The Minimum or Maximum of a Quadratic Function

Consider the quadratic function $f(x) = ax^2 + bx + c$, $a \neq 0$.

(1) If $a > 0$, then the parabola opens upward and f has a minimum $f\left(-\dfrac{b}{2a}\right)$ at the vertex.

(2) If $a < 0$, then the parabola opens downward and f has a maximum $f\left(-\dfrac{b}{2a}\right)$ at the vertex.

13.4 Intercepts of a Quadratic Function

Consider the quadratic function $f(x) = ax^2 + bx + c$, $a \neq 0$.

(1) The y-intercept is $(0, f(0)) = (0, c)$.

(2) The x-intercepts, if exist, are the solutions of the equation $ax^2 + bx + c = 0$.

Example 13.1. Does the function $f(x) = 2x^2 - 4x - 6$ have a maximum or minimum? Find it.

Example 13.2. Consider the function $f(x) = -x^2 + 3x + 6$. Find values of x such that $f(x) = 2$.

Example 13.3. A quadratic function f whose the vertex is $(1, 2)$ has a y-intercept $(0, -3)$. Find the equation that defines the function.

13.5 Practice

Exercise 13.1. Sketch the graph of the quadratic functions $f(x) = -(x-2)^2 + 4$ and find

(1) the coordinates of the x-intercepts,

(2) the coordinates of the y-intercept,

(3) the equation of the axis of symmetry,

(4) the coordinates of the vertex.

(5) the interval of x values such that $f(x) \geq 0$.

Exercise 13.2. Sketch the graph of the quadratic functions $f(x) = x^2 + 2x - 3$ and find

(1) the coordinates of the x-intercepts,

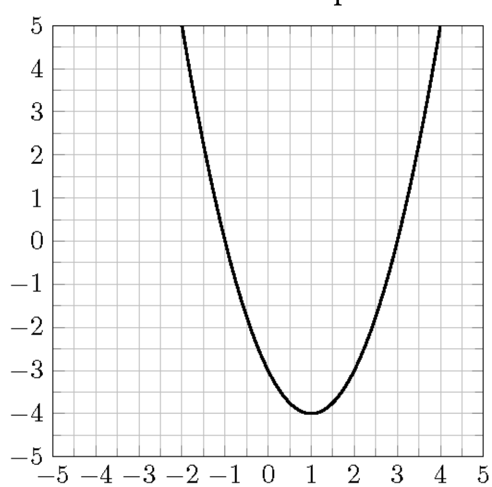
(2) the coordinates of the y-intercept,

(3) the equation of the axis of symmetry,

(4) the coordinates of the vertex.

(5) the interval of x values such that $f(x) > 0$.

Exercise 13.3. Consider the parabola in the graph.



(1) Determine the coordinates of the x-intercepts.

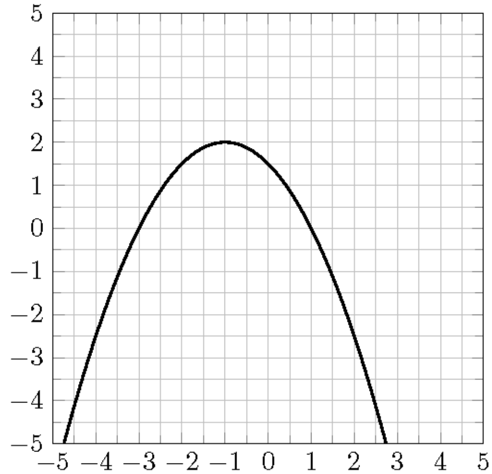
(2) Determine the coordinates of the y-intercept.

(3) Determine the coordinates of the vertex.

(4) For what values of x is $f(x) = -3$.

(5) Find an equation for the function.

Exercise 13.4. Consider the parabola in the graph.



(1) Determine the coordinates of the x-intercepts.

(2) Determine the coordinates of the y-intercept.

(3) Determine the coordinates of the vertex.

(4) For what values of x is $f(x) = \frac{3}{2}$.

(5) Find an equation for the function.

Exercise 13.5. A store owner estimates that by charging x dollars each for a certain cell phone case, he can sell $d(x) = 40 - x$ phone cases each week. The revenue in dollars is $R(x) = xd(x)$ when the selling price of a cell phone case is x, Find the selling price that will maximize revenue, and then find the amount of the maximum revenue.

Exercise 13.6. A ball is thrown upwards from a rooftop. It will reach a maximum vertical height and then fall back to the ground. The height $h(t)$ of the ball from the ground after time t seconds is $h(t) = -16t^2 + 48t + 160$ feet. How long it will take the ball to hit the ground?

Exercise 13.7. A ball is thrown upward from the ground with an initial velocity v_0 ft/sec. The height $h(t)$ of the ball after t seconds is $h(t) = -16t^2 + v_0 t$. The ball hits the ground after 4 seconds. Find the maximum height and how long it will take the ball to reach the maximum height.

Exercise 13.8. A toy factory estimates that the demand of a particular toy is $300 - x$ units each week if the price is $\$x$ dollars per unit. Each week there is a fixed cost \$40,000 to produce the demanded toys. The weekly revenue is a function of the price given by $R(x) = x(30 - x)$

(1) Find the function that models the weekly revenue, R, received when the selling price is $\$x$ per unit.

(2) What the price range so the the revenue is nonnegative.

14 Rational Functions

14.1 The Domain of a Rational Function

A ***rational function*** f is defined by an equation $f(x) = \frac{p(x)}{q(x)}$, where $p(x)$ and $q(x)$ are polynomials and the degree of $q(x)$ is at least one. Since the denominator cannot be zero, the domain of f consists all real numbers except the numbers such that $q(x) = 0$

Example 14.1. Find the domain of the function $f(x) = \frac{1}{x-1}$.

14.2 Practice

Exercise 14.1. Find the domain of each function. Write in interval notation.

(1) $f(x) = \frac{x^2}{x-2}$

(2) $f(x) = \frac{x}{x^2-1}$

15 Radical Functions

15.1 The Domain of a Radical Function

A ***radical function*** f is defined by an equation $f(x) = \sqrt[n]{r(x)}$, where $r(x)$ is an algebraic expression. For example $f(x) = \sqrt{x+1}$. When n is odd number, $r(x)$ can be any real number. When n is even, $r(x)$ has to be nonnegative, that is $r(x) \geq 0$ so that $f(x)$ is a real number.

Example 15.1. Find the domain of the function $f(x) = \sqrt{x+1}$.

15.2 Practice

Exercise 15.1. Find the domain of each function. Write in interval notation.

(1) $f(x) = \sqrt{1-x^2}$

(2) $f(x) = -\sqrt{\frac{1}{x-5}}$

16 Exponential Functions

16.1 Half-life

Half-life is the time required for a quantity to reduce to half of its initial value.

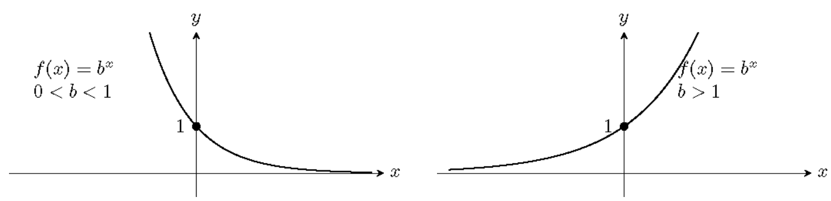
A certain pesticide is used against insects. The half life of the pesticide is about 12 days. After a month how much would left if the initial amount of the pesticide is 10 g? Can you write a function for the remaining quantity of the pesticide after t days?

16.2 Definition and Graphs of Exponential Functions

Let b be a positive number other than 1 (i.e. $b > 0$ and $b \neq 1$). The exponential function f of x with the base b is defined as

$$f(x) = b^x \qquad \text{or} \qquad y = b^x.$$

Graphs of exponential functions:



Remark. The exponential function $f(x) = b^x$ is an one-to-one function: any vertical line or any horizontal line crosses the graph at most once. Equivalently, the equation $b^x = c$ has at most one solution for any real number c.

16.3 The Natural Number e

The natural number e is the number to which the quantity $\left(1 + \frac{1}{n}\right)^n$ approaches as n takes on increasingly large values. Approximately, $e \approx 2.718281827$.

16.4 Compound Interests

After t years, the balance A in an account with a principal P and annual interest rate r is given by the following formulas:

(1) For n compounding periods per year: $A = P\left(1 + \frac{r}{n}\right)^{nt}$.

(2) For compounding continuously: $A = Pe^{rt}$.

Example 16.1. A sum of $\$10,000$ is invested at an annual rate of 8%, Find the balance, to the nearest hundredth dollar, in the account after 5 years if the interest is compounded

(1) monthly,

(2) quarterly,

(3) semiannually,

(4) continuous.

Remark. In the compounded investment module, the $\frac{r}{n}$ is an approximation of the period interest rate. Indeed, if the period rate p satisfies the equation $(1 + p)^n = 1 + r$, or equivalently $p = \sqrt[n]{(1+r)} - 1$. Using the formula $(1+x)^n = 1+nx+\frac{n(n-1)}{2}x^2+\cdots+x^n$, one may approximately replace $1 + r$ by $(1 + \frac{r}{n})$ and obtain the approximation $p \approx \frac{r}{n}$.

Example 16.2. The population of a country was about 0.78 billion in the year 2015, with an annual growth rate of about 0.4%. The predicted population is $P(t) = 0.78(1.004)^t$ billions after t years since 2015. To the nearest thousandth of a billion, what will the predicted population of the country be in 2030?

16.5 Practice

Exercise 16.1. The value of a car is depreciating according to the formula: $V = 25000(3.2)^{-0.05x}$, where x is the age of the car in years. Find the value of the car, to the nearest dollar, when it is five years old.

Exercise 16.2. A sum of $20,000 is invested at an annual rate of 5.5%, Find the balance, to the nearest dollar, in the account after 5 years subject to

(1) monthly compounding,

(2) continuously compounding.

Exercise 16.3. Sketch the graph of the function and find its range.

(1) $f(x) = 3^x$

(2) $f(x) = \left(\frac{1}{3}\right)^x$

Exercise 16.4. Use the given function to compare the values of $f(-1.05)$, $f(0)$ and $f(2.4)$ and determine which value is the largest and which value is the smallest. Explain your answer.

(1) $f(x) = \left(\frac{5}{2}\right)^x$

(2) $f(x) = \left(\frac{2}{3}\right)^x$

17 Logarithmic Functions

17.1 Estimate the Number of Digits

Can you estimate the number of digits in the integer part of the number $2^{15} \times \sqrt{2020} \div 2021$?

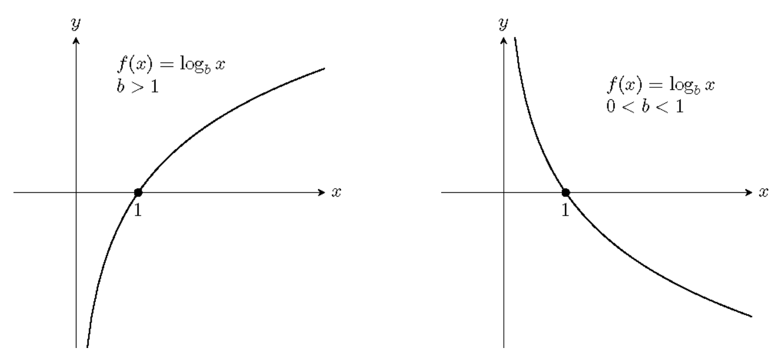
17.2 Definition and Graphs of Logarithmic Function

For $x > 0$, $b > 0$ and $b \neq 1$, there is a unique number y satisfying the equation $b^y = x$. We denote the unique number y by $\log_b x$, read as logarithm to the base b of x. In other words, the defining relation between exponentiation and logarithm is

$$y = \log_b x \quad \text{if and only if} \quad b^y = x.$$

The function $f(x) = \log_b x$ is called the logarithmic function f of x with the base b.

Graphs of logarithmic functions:



17.3 Commonand Natural Logarithms

A logarithmic function $f(x)$ with base 10 is called the common logarithmic function and denoted by $f(x) = \log x$.

A logarithmic function $f(x)$ with base the natural number e is called the natural logarithmic function and denoted by $f(x) = \ln x$.

17.4 Basic Properties of Logarithms

When $b > 0$ and $b \neq 1$, and $x > 0$, we have

(1) $b^{\log_b x} = x$.

(2) $\log_b(b^x) = x$.

(3) $\log_b b = 1$ and $\log_b 1 = 0$.

Example 17.1. Convert between exponential and logarithmic forms.

(1) $\log x = \frac{1}{2}$

(2) $3^{2x-1} = 5$

Example 17.2. Evaluate the logarithms.

(1) $\log_4 2$

(2) $10^{\log(\frac{1}{2})}$

(3) $\log_5(e^0)$

Example 17.3. Find the domain of the function $f(x) = \ln(2 - 3x)$.

17.5 Properties of Logarithms

For $M > 0$, $N > 0$, $b > 0$ and $b \neq 1$, we have

(1) (The product rule) $\log_b(MN) = \log_b M + \log_b N$

(2) (The quotient rule) $\log_b(\frac{M}{N}) = \log_b M - \log_b N$.

(3) (The power rule) $\log_b(M^p) = p \log_b M$, where p is any real number.

(4) (The change-of-base property) $\log_b M = \dfrac{\log_a M}{\log_a b}$, where $a > 0$ and $a \neq 1$. In particular,

$$\log_b M = \frac{\log M}{\log b} \quad \text{and} \quad \log_b M = \frac{\ln M}{\ln b}.$$

Example 17.4. Expand and simplify the logarithm $\log_2\left(\frac{8\sqrt{y}}{x^3}\right)$.

Example 17.5. Write the expression $2\ln(x-1) - \ln(x^2+1)$ as a single logarithm.

Example 17.6. Evaluate the logarithm $\log_3 4$ and round it to the nearest tenth.

Example 17.7. Simplify the logarithmic expression

$$\log_2(x^{\log 3}) \log_3 2.$$

17.6 Practice

Exercise 17.1. Write each equation into equivalent exponential form.

(1) $\log_3 7 = y$

(2) $3 = \log_b 64$

(3) $\log x = y$

(4) $\ln(x-1) = c$

Exercise 17.2. Write each equation into equivalent logarithmic form.

(1) $7^x = 10$

(2) $b^5 = 2$

(3) $e^{2y-1} = x$

(4) $10^x = c^2 + 1$

Exercise 17.3. Evaluate.

(1) $\log_2 16$

(2) $\log_9 3$

(3) $\log 10$

(4) $\ln 1$

Exercise 17.4. Evaluate.

(1) $e^{\ln 2}$

(2) $\log 10^{\frac{1}{3}}$

(3) $\ln(\sqrt{e})$

(4) $\log_2(\frac{1}{2})$

Exercise 17.5. Find the domain of the function $f(x) = \log(x - 5)$. Write in interval notation.

Exercise 17.6. Sketch the graph of each function and find its range.

(1) $f(x) = \log_2 x$

(2) $f(x) = \log_{\frac{1}{2}} x$

Exercise 17.7. Expand the logarithm and simplify.

(1) $\log(100x)$

(2) $\ln\left(\frac{10}{e^2}\right)$

(3) $\log_b(\sqrt[3]{x})$

(4) $\log_7(\frac{x^2\sqrt{y}}{z})$

Exercise 17.8. Expand the logarithm and simplify.

(1) $\log_b \sqrt{\frac{x^2 y}{5}}$

(2) $\ln(\sqrt[3]{(x^2+1)y^{-2}})$

(3) $\log(x\sqrt{10x} - \sqrt{10x})$

Exercise 17.9. Write as a single logarithm.

(1) $\frac{1}{3}\log x + \log y$

(2) $\frac{1}{2}\ln(x^2+1) - 2\ln x$

(3) $\frac{1}{3}\log_2 x - 3\log_2(x+1) + 1$

Exercise 17.10. Write as a single logarithm.

(1) $2\log(2x+1) - \frac{1}{2}\log x$

(2) $3\ln x - 5\ln y + \frac{1}{2}\ln z$

(3) $3\log_3 x - 2\log_3(1-x) + \frac{1}{3}\log_3(x^2+1)$.

Exercise 17.11. Evaluate the logarithm and round it to the nearest hundredth.

(1) $\log_2 10$

(2) $\log_3 5$

(3) $\dfrac{1}{\log_5 2}$

(4) $\log_4 5 - \log_2 9$

Exercise 17.12. Simplify the logarithmic expression

$$\frac{\log_3(x^2)\log_y\sqrt{3}}{\log x}.$$

18 Applications of Exp and Log Functions

18.1 Newton's Law of Cooling

Suppose an object with an initial temperature $T(0)$ is placed in an environment with surrounding temperature T_{env}. By Newton's Law of Cooling, after t minutes, the temperature of the object $T(t)$ is given by the exponential function

$$T(t) = T_{\text{env}} + (T(0) - T_{\text{env}})\, e^{-rt},$$

where r is a positive constant characteristic of the system.

A cup of coffee is brewed with a temperature 195°F and placed in a room with the temperature 60°F. The cooling constant for a cup of coffee is $r = 0.09\text{min}^{-1}$.

(1) After 30 minutes, what is the temperature of the coffee?

(2) How long it takes for the coffee to cool down to the room temperature?

18.2 Exponential and Logarithmic Equations

To solve an exponential or logarithmic equation, the first step is to rewrite the equation with a single exponentiation or logarithm. Then we can use the equivalent relation between exponentiation and logarithm to rewrite the equation and solve the resulting equation.

Example 18.1. Solve the equation $10^{2x-1} - 5 = 0$.

Example 18.2. Solve the equation $\log_2 x + \log_2(x-2) = 3$.

18.3 Solving Compound Interest Model

Example 18.3. A check of \$5000 was deposited in a savings account with an annual interest rate 6% which is compounded monthly. How many years will it take for the money to raise by 20%?

18.4 Practice

Exercise 18.1. Solve the exponential equation.

(1) $2^{x-1} = 4$

(2) $7e^{2x} - 5 = 58$

Exercise 18.2. Solve the exponential equation.

(1) $3^{x^2-2x} = e^{-\ln 3}$

(2) $2^{(x+1)} = 3^{(1-x)}$

Exercise 18.3. Solve the logarithmic equation.

(1) $\log_5 x + \log_5(4x - 1) = 1$

(2) $\ln \sqrt{x+1} = 1$

Exercise 18.4. Solve the logarithmic equation.

(1) $\log_2(x + 2) - \log_2(x - 5) = 3$

(2) $\log_3(x-5) = 2 - \log_3(x+3)$

Exercise 18.5. For the given function, find values of x satisfying the given equation.

(1) $f(x) = \log_4 x - 2\log_4(x+1), \quad f(x) = -1$

(2) $g(x) = \log(2-5x) + \log(-x), \quad g(x) = 1$

Exercise 18.6. Find intersections of the given pairs of curves.

(1) $f(x) = e^{x^2}$ and $g(x) = e^x + 12$.

(2) $f(x) = \log_7\left(\frac{1}{2}(x+2)\right)$ and $g(x) = 1 - \log_7(x-3)$

Exercise 18.7. Using the formula $A = P(1+\frac{r}{n})^{nt}$ to determine how many years, to the nearest hundredth, it will take to double an investment \$20,000 at the interest rate 5% compounded monthly.

Exercise 18.8. Newton's Law of Cooling states that the temperature T of an object at any time t satisfying the equation $T = T_s + (T_0 - T_s)e^{-rt}$, where T_s is the the temperature of the surrounding environment, T_0 is the initial temperature of the object, and r is positive constant characteristic of the system, which is in units of time^{-1}. In a room with a temperature of $22°C$, a cup of tea of $97°C$ was freshly brewed. Suppose that $r = \ln 5/20$ minute^{-1}. In how many minutes, the temperature of the tea will be $37°C$?

19 Methods to Solve a Linear Systems

A ***system of linear equations*** of two variables consists of two equations. A ***solution of a system*** of linear equations of two variables is an ordered pair that satisfies both equations.

19.1 Substitution Method

Example 19.1. Solve the system of linear equations using the substitution method.

$$\begin{aligned} x + y &= 3 \\ 2x + y &= 4 \end{aligned}$$

19.2 Elimination Method

Example 19.2. Solve the system of linear equations using the addition method.

$$\begin{aligned} 5x + 2y &= 7 \\ 3x - y &= 13 \end{aligned}$$

Remark. A linear system may have ***one solution***, ***no solution*** or ***infinitely many solutions***.

If the lines defined by equations in the linear system have the same slope but different y-intercepts or the elimination method ends up with $0 = c$, where c is a nonzero constant, then the system has no solution.

If the lines defined by equations in the linear system have the same slope and the same y-intercept or the elimination method ends up with $0 = 0$, then the system has infinitely many solutions. In this case, we say that the system is ***dependent*** and a solution can be expressed in the form $(x, f(x)) = (x, mx + b)$.

19.3 Practice

Exercise 19.1. Solve.

$$\begin{aligned} 2x - y &= 8 \\ -3x - 5y &= 1 \end{aligned}$$

Exercise 19.2. Solve.

$$\begin{aligned} x + 4y &= 10 \\ 3x - 2y &= -12 \end{aligned}$$

Exercise 19.3. Solve.

$$\begin{aligned} -x - 5y &= 29 \\ 7x + 3y &= -43 \end{aligned}$$

Exercise 19.4. Solve.

$$\begin{aligned} 2x + 15y &= 9 \\ x - 18y &= -21 \end{aligned}$$

Exercise 19.5. Solve.

$$\begin{aligned} 2x + 7y &= 5 \\ 3x + 2y &= 16 \end{aligned}$$

Exercise 19.6. Solve.

$$\begin{aligned} 4x + 3y &= -10 \\ -2x + 5y &= 18 \end{aligned}$$

Exercise 19.7. Solve.

$$\begin{aligned} 3x + 2y &= 6 \\ 6x + 4y &= 16 \end{aligned}$$

Exercise 19.8. Solve.

$$\begin{aligned} 2x - 3y &= -6 \\ -4x + 6y &= 12 \end{aligned}$$

Exercise 19.9. Last week, Mike got 5 apples and 4 oranges for \$7. The week the prices are still the same and he got 3 apples and 6 oranges for \$6. What's the price for 1 apple and 1 orange?

Exercise 19.10. The sum of the digits of a certain two-digit number is 7. Reversing its digits increases the number by 27.
What is the number?