MA440 Precalculus Worksheet

Fei Ye

Department of Mathematics and Computer Science
Queensborough Community College - CUNY
2024

Last updated on March 18, 2024

Dr. Fei Ye
Department of Mathematics and Computer Science
Queensborough Community College of CUNY
222-05 56th Street, Bayside, NY, 11364
email: feye@qcc.cuny.edu

MA440 Precalculus Worksheet

Preface

Those worksheets are developed for the Precalculus course at QCC. Contents in those worksheets are mainly based on the OpenStax Precalculus textbook.

Contents

1.1 Basic Concepts

Definition 1.1.1 A **relation** is a set of ordered pairs. The set of the first components of each ordered pair is called the **domain** and the set of the second components of each ordered pair is called the **range**.

A **function** is a relation that assigns each element in the domain a unique element in the range.

An arbitrary value in the domain is often represented by the lowercase letter x which is called an **independent variable**. An arbitrary output is often represented by the lowercase letter y which is called a **dependent variable**.

Example 1.1.1 Determine if the relation

$$\{(1,2),(2,4),(3,6),(4,8),(5,10)\}$$

is a function. Find the domain and the range.

Definition 1.1.2 If a function has x as the independent variable and y as the dependent variable, then we often say that y is a function of x.

Example 1.1.2 Consider items and prices in a grocery store. Is price a function of item? Is item a function of price?

Definition 1.1.3 A function is often named by letters, such as f, F, p, or q. If f is a function of x, then we denote it as $y = f(x)$ which is called the **function notation**. Here $f(x)$ is read as f of x or f at x. The notation $f(x)$ represents the output of the function f for a given input x.

Example 1.1.3 Use function notation to represent a function whose input is the name of a month and output is the number of days in that month.

Example 1.1.4 A function $N = f(y)$ gives the number of police officers, N, in a town in year y. What does $f(2005) = 300$ represent?

Example 1.1.5 Using a table to represent the days in the month as the function of month.

Example 1.1.6 Consider the function $f(x) = x^2 + 3x - 4$. Find the values of the following expressions.

(1) $f(2)$ (2) $f(a)$ (3) $f(a+h)$ (4) $\dfrac{f(a+h) - f(a)}{h}$

Example 1.1.7 Consider the function $f(x) = x^2 - 2x$. Find all x values such that $f(x) = 3$.

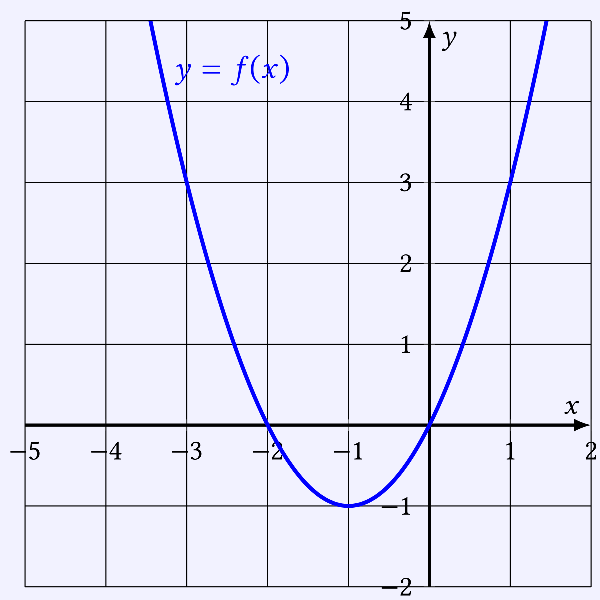
Example 1.1.8 Express the relationship defined by the function $2x - y - 3 = 0$ as a function $y = l(x)$.

Example 1.1.9 Does the equation $x^2 + y^2 = 1$ defines y as a function x. If so, express the relationship as a function $y = f(x)$. If not, under what extra condition does the function $y = f(x)$ exist?

Example 1.1.10 Consider the function $f(x)$ defined by a graph below.

(1) Find $f(-1)$.

(2) Find all x such that $f(x) = 3$.



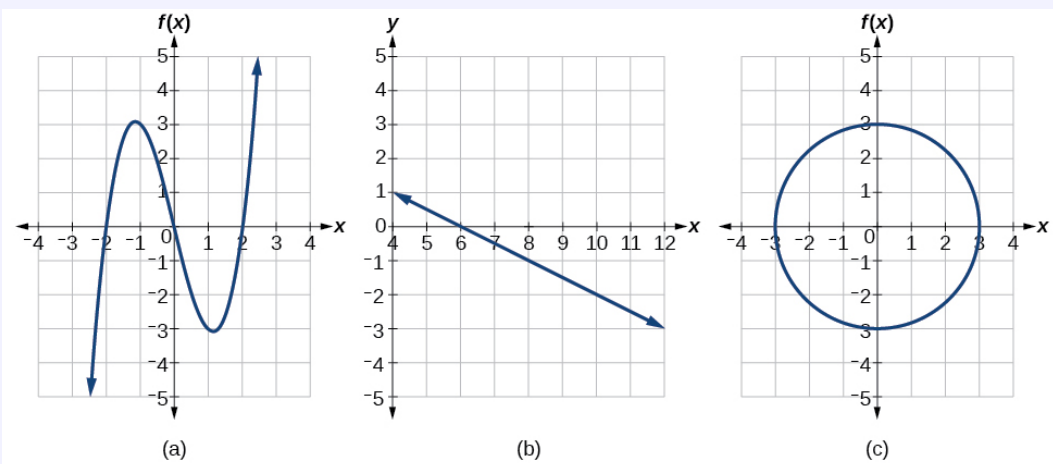
Definition 1.1.4 A function is a **one-to-one function** if each output value corresponds to exactly one input value.

Example 1.1.11 Is the area of a circle a function of its radius? If yes, is the function one-to-one?

How-to *A graph is a function if very vertical line crosses the graph at most once. This method is known as the* ***vertical line test****.*

A function is an one-to-one if very horizontal line crosses the graph at most once. This method is known as the ***horizontal line test****.*

Example 1.1.12 Determine if the graph defines a function. If so, is it a one-to-one function?



Exercises

✍ **Exercise 1.1.1** Consider the function $f(x) = 2x^2 + x - 3$. Find the values of the following expressions.

(1) $f(-1)$ (2) $f(a)$ (3) $f(a+h)$ (4) $\dfrac{f(a+h) - f(a)}{h}$

✍ **Exercise 1.1.2** For the function $f(x) = -4x + 5$, evaluate and simplify the difference quotient $\dfrac{f(x+h) - f(x)}{h}$.

✍ **Exercise 1.1.3** Consider the function $f(x) = -x^2 - 4x$. Find all x values such that $f(x) = 3$.

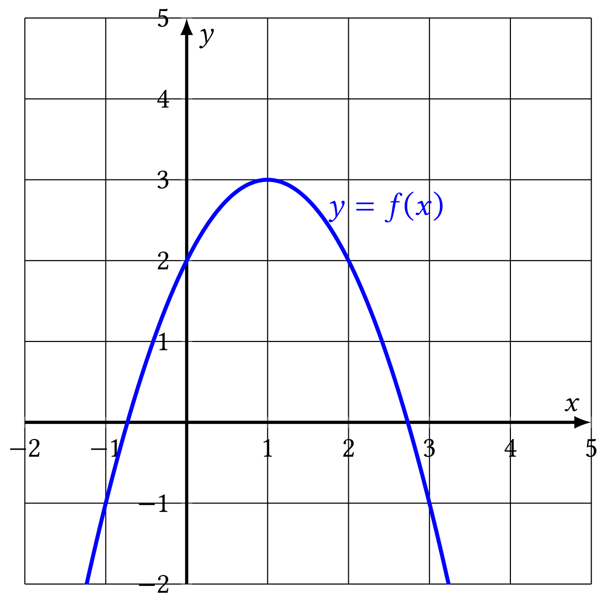
Exercise 1.1.4 Express the relationship defined by the function $3x - 2y - 6 = 0$ as a function $y = l(x)$.

Exercise 1.1.5 If $8x - y^3 = 0$, express y as a function of x.
Is y a one-to-one function of x?

Exercise 1.1.6 Consider the function $f(x)$ defined by a graph below.

(1) Find $f(1)$.

(2) Find all x such that $f(x) = 3$.



1.2 Domains and Ranges

How-to *The domain of a function f consists of possible input values x. Or equivalently, the domain consists of all x values except those that will make the function is undefined.*

The range of a function f consists of all possible output values y. Equivalently, the range consists of y value such that equation $y = f(x)$ has a solution x.

Example 1.2.1 Find the domain of the function

$$f(x) = \frac{x+1}{2-x}.$$

Example 1.2.2 Find the domain of the function

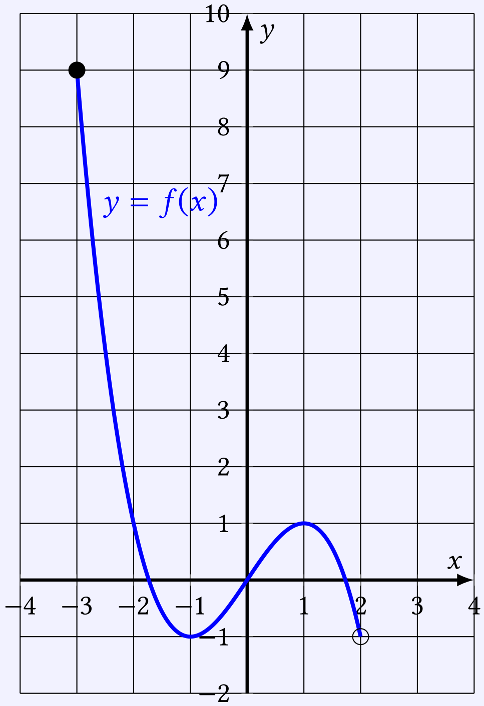
$$f(x) = \sqrt{7-x}.$$

Definition 1.2.1 **Set-builder notation** is a method of specifying a set of elements that satisfy a certain condition. It takes the form $\{x|$ statement about x$\}$ which is read as, “the set of all x such that the statement about x is true."

Interval notation is a way of describing sets that include all real numbers between a lower limit that may or may not be included and an upper limit that may or may not be included. The endpoint values are listed between brackets or parentheses. A square bracket indicates inclusion in the set, and a parenthesis indicates exclusion from the set.

Example 1.2.3 Find the domain of the function $f(x) = \dfrac{\sqrt{x+2}}{x-1}$. Write your answer in set-builder notation and interval notation.

Example 1.2.4 Find the domain and range of the function f whose graph is shown in Figure.



Example 1.2.5 Find the domain and range of the function

$$f(x) = \frac{2}{x+3}.$$

Example 1.2.6 Find the domain and range of the function

$$f(x) = 3\sqrt{x+2}.$$

Example 1.2.7 Consider the piecewise function

$$f(x) = \begin{cases} 2x-3 & \text{if} \quad x \leq -1 \\ -x^2 & \text{if} \quad -1 < x < 1 \\ -2x+4 & \text{if} \quad 1 \leq x. \end{cases}$$

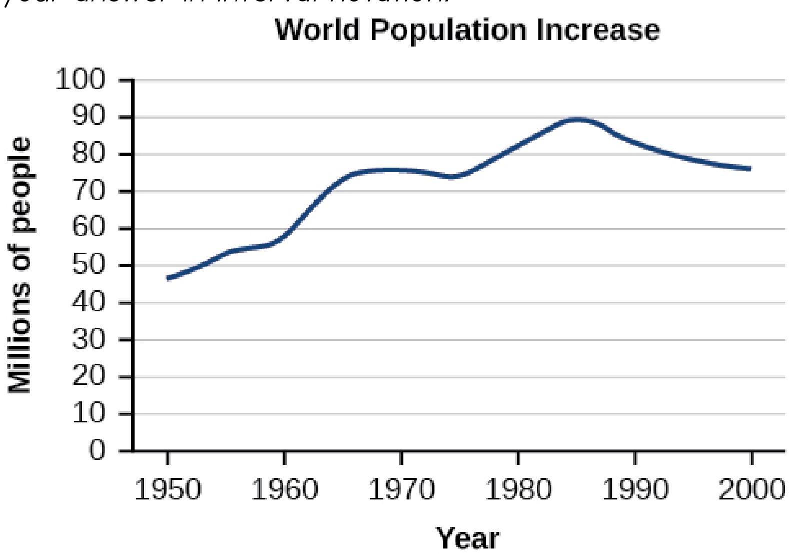
(1) Sketch the graph (2) Find $f(-4)$ (3) Find $f(2)$

Exercises

✍ **Exercise 1.2.1** Find the domain of the function

(1) $f(x) = \dfrac{1+4x}{2x-1}$ (2) $f(x) = \sqrt{5+2x}$ (3) $f(x) = \dfrac{\sqrt{x+1}}{x-1}$ (4) $f(x) = \dfrac{x-2}{x^2+7x-44}$

✍ **Exercise 1.2.2** Estimate the domain and range for the function defined by the graph. Write your answer in interval notation.



Exercise 1.2.3 Find the domain and range of each of the following functions. Write your answer in set-builder notation and interval notation.

(1) $f(x) = \frac{3}{x-2}$ (2) $f(x) = -2\sqrt{x+4}$

Exercise 1.2.4 Consider the piecewise function

$$f(x) = \begin{cases} -2x+5 & \text{if} \quad x < -2 \\ x^2 - 1 & \text{if} \quad -2 \leq x \leq 2 \\ 2x - 3 & \text{if} \quad 2 < x. \end{cases}$$

(1) Sketch the graph (2) Find $f(-4)$ (3) Find $f(2)$

1.3 Rates of Change and Behavior of Graphs

Definition 1.3.1 (Rate of Change) The average rate of change of f over an interval $[a, b]$ is defined as

$$\text{Average Rate Of Change} = \frac{f(b) - f(a)}{b - a}.$$

The average rate of change is the same as the slope of secant line passing through $(a, f(a))$ and $(b, f(b))$.

By taking $x = a$ and $h = b - a$, the average of rate of change is the same the difference quotient of a function f which is defined as

$$\text{Difference Quotient} = \frac{f(x + h) - f(x)}{h}.$$

Example 1.3.1 After picking up a friend who lives 10 miles away, Anna records her distance from home over time. The values are shown in Table. Find her average speed over the first 6 hours.

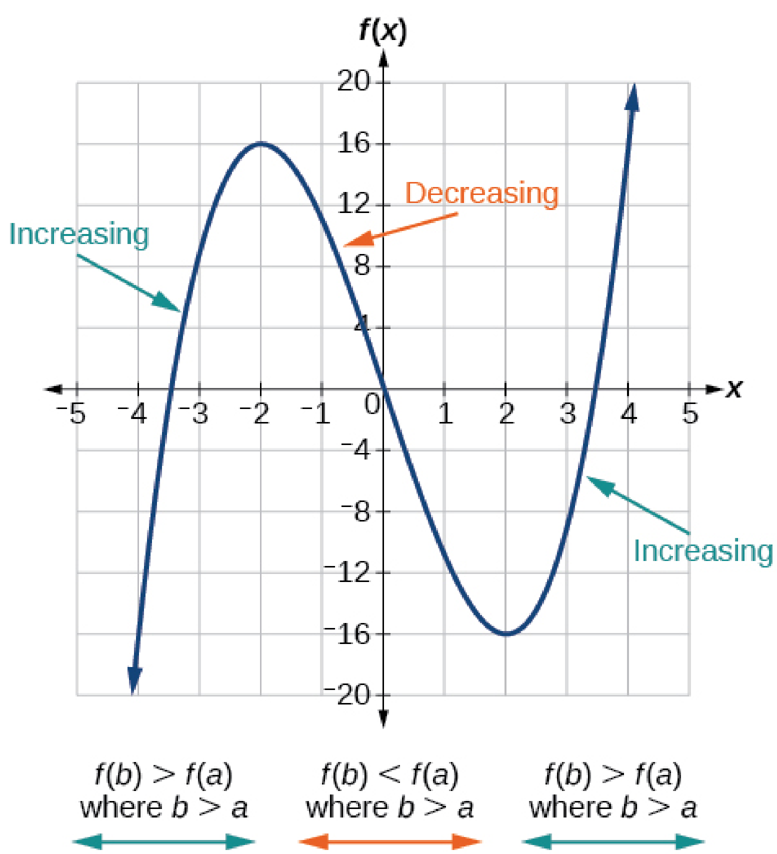
t (hours)	0	1	2	3	4	5	6	7
$D(t)$ (miles)	10	55	90	153	214	240	292	300

Example 1.3.2 Find the average rate of change of $f(x) = x^2 - \dfrac{1}{x}$ over the interval $[2, 4]$.

Example 1.3.3 Find the average rate of change of $g(t) = t^2 + 3t + 1$ on the interval $[0, a]$. The answer will be an expression involving a.

Definition 1.3.2 (Increasing and Decreasing) A function f is **increasing** over an interval (a, b) if $f(x_2) > f(x_1)$ for any $x_1 < x_2$ in (a, b). Equivalently, f is increasing over (a, b) if the average rate of change is positive over any subinterval (x_1, x_2) of (a, b).

A function f is **decreasing** over an interval (a, b) if $f(x_2) < f(x_1)$ for any $x_1 < x_2$ in (a, b). Equivalently, f is decreasing over (a, b) if the average rate of change is negative over any subinterval (x_1, x_2) of (a, b).



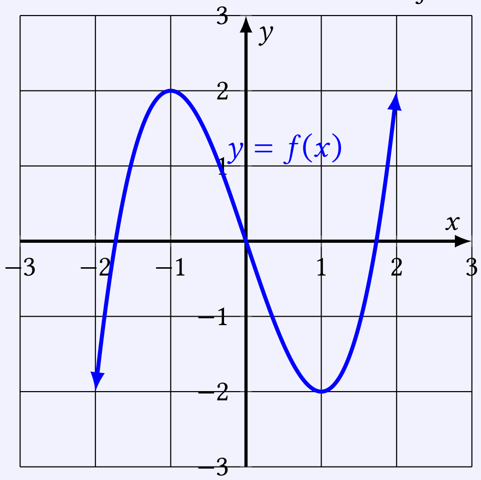
Definition 1.3.3 (Local Maxima and Minima) A function f has a **local maximum** at $x = c$ if $f(c) \geq f(x)$ for any x in a small interval containing c. A small interval containing c is also known as a small neighborhood of c.

A function f has a **local minimum** at $x = c$ if $f(c) \leq f(x)$ for any x in a small interval containing c.

How-to *A function f has a local maximum at $x = c$ if it changes from increasing to decreasing at c in a neighborhood of c.*

A function f has a local minimum at $x = c$ if it changes from decreasing to increasing at c in a neighborhood of c.

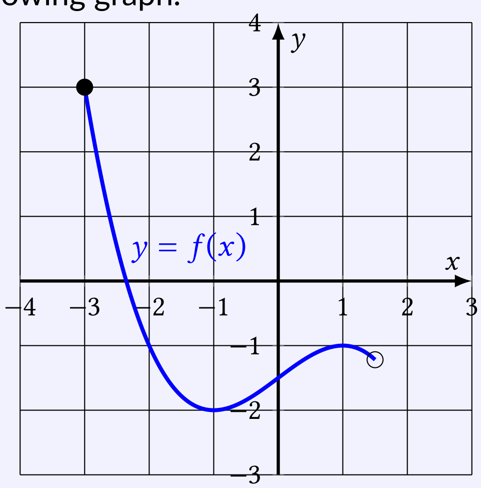
Example 1.3.4 Find the interval of increasing and the interval of decreasing, and the local maxima and local minima of the function f defined by the following graph.



Definition 1.3.4 (Absolute Maxima and Minima) The **absolute maximum** of f at $x = c$ is $f(c)$ where $f(c) \geq f(x)$ for all x in the domain of f.

The **absolute minimum** of f at $x = c$ is $f(c)$ where $f(c) \leq f(x)$ for all x in the domain of f.

Example 1.3.5 Finding the absolute maximum and minimum of the function f defined by the following graph.

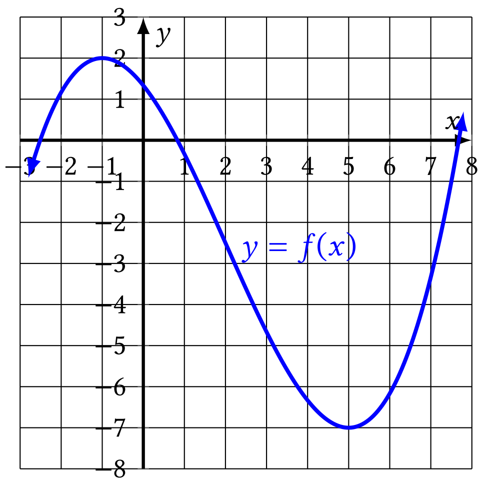


Exercises

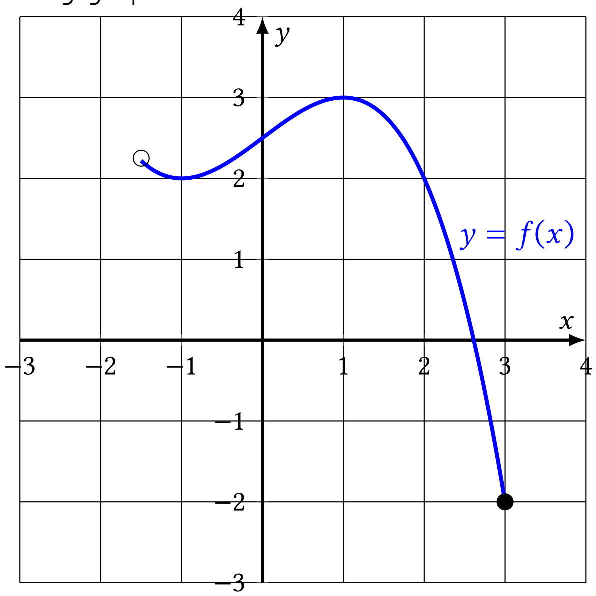
✍ **Exercise 1.3.1** The electrostatic force F, measured in newtons, between two charged particles can be related to the distance between the particles d, in centimeters, by the formula $F(d) = \frac{2}{d^2}$. Find the average rate of change of force if the distance between the particles is increased from 2 cm to 6 cm.

✍ **Exercise 1.3.2** Find the average rate of change of $f(x) = x^2 + 2x - 8$ on the interval $[5, a]$.

✍ **Exercise 1.3.3** Find the interval of increasing and the interval of decreasing, and the local maxima and local minima of the function f defined by the following graph.



✍ **Exercise 1.3.4** Finding the absolute maximum and minimum of the function f defined by the following graph.



✍ **Exercise 1.3.5** Find the interval of increasing and the interval of decreasing, and the local maxima and local minima of the function $f(x) = x^3 - 6x^2 - 15x + 20$ using its graph.

1.4 Combination and Composition of Functions

Definition 1.4.1 (Algebraic Operations of Functions) Let f and g be two functions with domains A and B respectively. We define the linear combination, product, and quotient functions as follows.

Linear combination:	$(af + bg)(x) = af(x) + bg(x)$	with the domain $A \cap B$.
Product:	$(fg)(x) = f(x)g(x)$	with the domain: $A \cap B$.
Quotient:	$\left(\dfrac{f}{g}\right)(x) = \dfrac{f(x)}{g(x)}$	with the domain: $A \cap B \cap \{x \mid g(x) \neq 0\}$.

Example 1.4.1 Consider the functions $f(x) = x - 1$ and $g(x) = x^2 - 1$. Find and simplify the functions $(g - f)(x)$ and $\left(\dfrac{g}{f}\right)(x)$, and their domains.

Definition 1.4.2 (Composition of functions) Let f and g be two functions with domains A and B respectively. The **composite function** $f \circ g$ (also called the composition of f and g) is defined as

$$(f \circ g)(x) = f(g(x)) \qquad \text{with the domain:} \quad B \cap \{x \mid g(x) \in A\}.$$

We read the left-hand side as "f composed with g at x," and the right-hand side as "f of g of x."

Example 1.4.2 Consider the functions $f(x) = \sqrt{x - 2}$ and $g(x) = x^2 + 1$.

(1) Find and simplify the functions $(f \circ g)(x)$ and $(g \circ f))(x)$. Are they the same function?

(2) Find the domains of $f \circ g$ and $g \circ f$. Are they the same?

Example 1.4.3 Consider $f(t) = t^2 - 4t$ and $h(x) = \sqrt{x+3}$. Evaluate

(1) $\dfrac{f(1)}{g(1)}$ (2) $h(f(-1))$ (3) $(f \circ h)(-1))$ (4) $(f - h)(-1)$

Example 1.4.4 Using the graphs to evaluate the given functions.

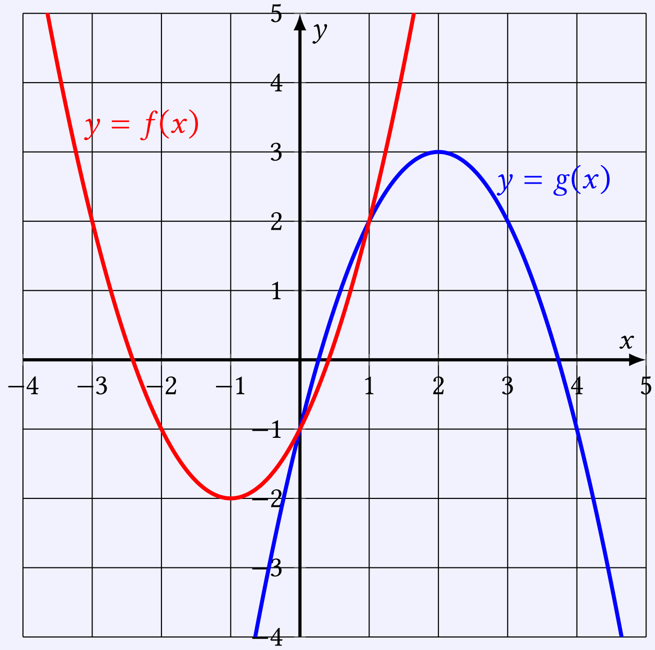
(1) $(f + g)(1)$

(2) $(fg)(1)$

(3) $\left(\dfrac{f}{g}\right)(1)$

(4) $(g \circ f)(-3)$

(5) $f(g(0))$



Example 1.4.5 Consider the function $h(x) = \sqrt{x^2+1}$. Find two functions f and g so that $h(x) = f(g(x))$.

Exercises

✍ **Exercise 1.4.1** Consider the functions $f(x) = x^2 - 1$ and $g(x) = x + 1$.

(1) Find the function $(f - g)(x)$ and its domain.

(2) Find the function $(fg)(x)$ and its domain.

(3) Find $\left(\dfrac{f}{g}\right)(x)$ and its domain.

(4) Find $(2f - 3g)(1)$.

(5) Find $2fg - \left(\dfrac{3g}{f}\right)(1)$.

✍ **Exercise 1.4.2** Consider the functions $f(x) = \dfrac{1}{x-2}$ and $g(x) = \sqrt{x+4}$.

(1) Find $f \circ g$ and its domain.

(2) Find $(g \circ f)(3)$.

Exercise 1.4.3 Using the graphs to evaluate the given functions.

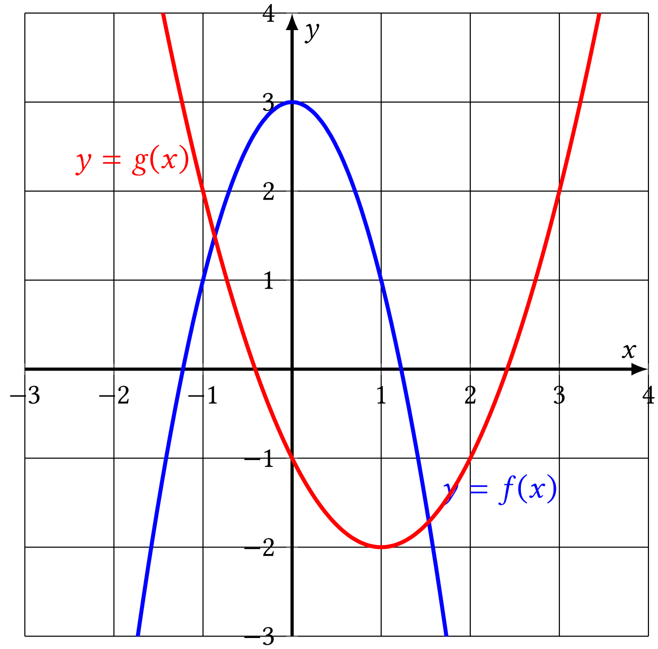
(1) $(f-g)(1)$

(2) $(fg)(0)$

(3) $\left(\frac{f}{g}\right)(0)$

(4) $(f \circ g)(2)$

(5) $g(f(0))$



Exercise 1.4.4 Consider the function $h(x) = \sqrt[3]{2x-1}$. Find two functions f and g so that $h(x) = f(g(x))$.

1.5 Transformations

Definition 1.5.1 Given a function $y = f(x)$, the function $y = f(x) + k$, where k is a constant, is a **vertical shift** of the function f.

How-to *Suppose k is positive.*
- *To graph $y = f(x)+k$, shift the graph of $y = f(x)$ upward k units.*
- *To graph $y = f(x)-k$, shift the graph of $y = f(x)$ downward k units.*

Example 1.5.1 Consider the functions $f(x) = x^2$, $g(x) = x^2 - 1$ and $h(x) = x^2 + 2$.

(1) Describe how to get the graph of g from the graph of f.

(2) Describe how to get the graph of h from the graph of f.

(3) Describe how to get the graph of f from the graph of h.

(4) Describe how to get the graph of h from the graph of g.

Definition 1.5.2 Given a function $y = f(x)$, the function $y = f(x - h)$, where h is a constant, is a **horizontal shift** of the function f.

How-to *Suppose h is positive.*
- *To graph $y = f(x-h)$, shift the graph of $y = f(x)$ to the right h units.*
- *To graph $y = f(x+h)$, shift the graph of $y = f(x)$ to the left h units.*

Example 1.5.2 Consider the functions $f(x) = x^2$, $g(x) = (x + 1)^2$ and $h(x) = (x - 2)^2$.

(1) Describe how to get the graph of g from the graph of f.

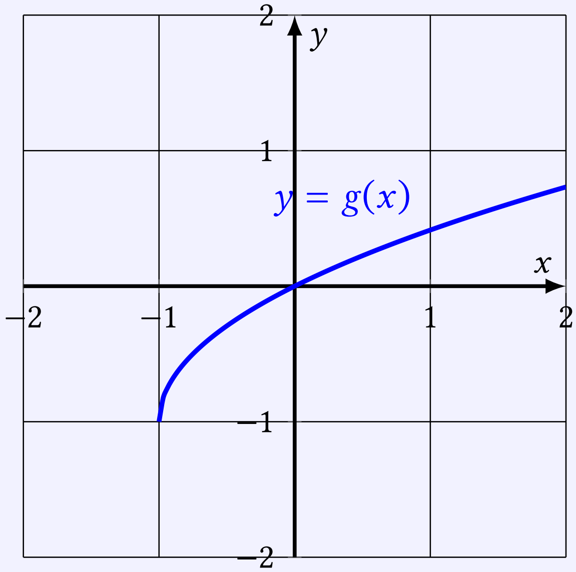
(2) Describe how to get the graph of h from the graph of f.

(3) Describe how to get the graph of f from the graph of h.

(4) Describe how to get the graph of h from the graph of g.

Example 1.5.3 Sketch the graph of $f(x) = |x|$. Then use the graph to sketch the graph of $h(x) = f(x+2) - 1$.

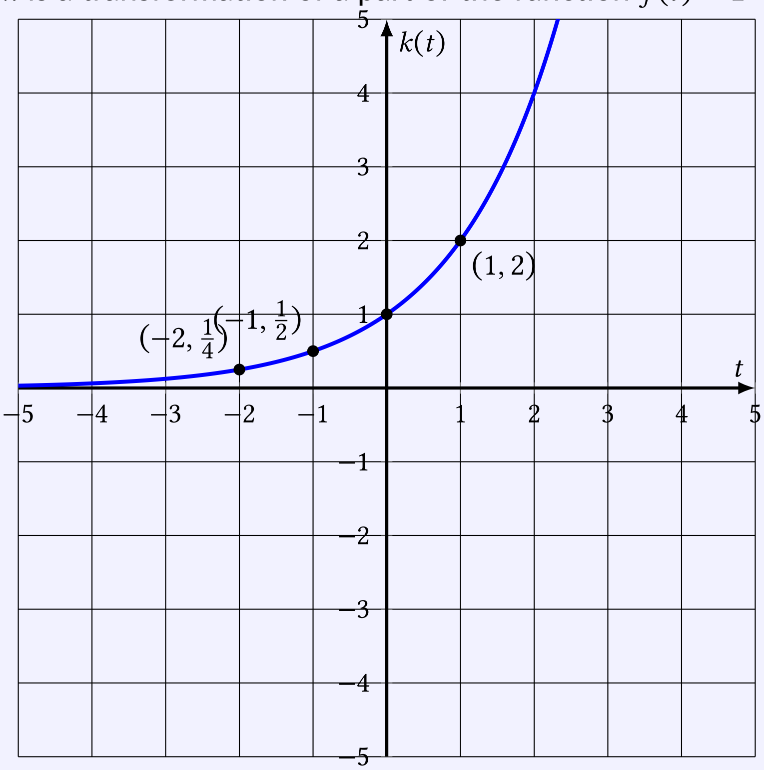
Example 1.5.4 The function $y = g(x)$ shown in the picture is a shift of the square root function $y = \sqrt{x}$. Find $g(x)$.



Definition 1.5.3 Given a function $y = f(x)$, the function $g(x) = -f(x)$ is a **vertical reflection** of the function $y = f(x)$, or a reflection about the x-axis; the function $g(x) = f(-x)$ is a **horizontal reflection** of the function $y = f(x)$ or a reflection about the y-axis.

Example 1.5.5 Reflect the graph of $f(x) = |x - 1|$
(1) first vertically, (2) then horizontally.
Denote the new function by $y = g(x)$. Find $g(x)$.

Example 1.5.6 A common model for learning has an equation similar to $k(t) = -2^{-t} + 1$, where k is the percentage of mastery that can be achieved after t practice sessions, and $t > 0$. The function k is a transformation of a part of the function $f(t) = 2^t$ shown below. Sketch the graph of $k(t)$.



Definition 1.5.4 A function is called an **even function** if $f(-x) = f(x)$ for x in the domain of f.
A function is called an **odd function** if $f(-x) = -f(x)$ for x in the domain of f.

Remark The graph of an even function is symmetric about y-axis.
The graph of an odd function is symmetric about the origin. This symmetry is known as a rotation symmetry.

Example 1.5.7 Group the functions according to even, odd, or other.

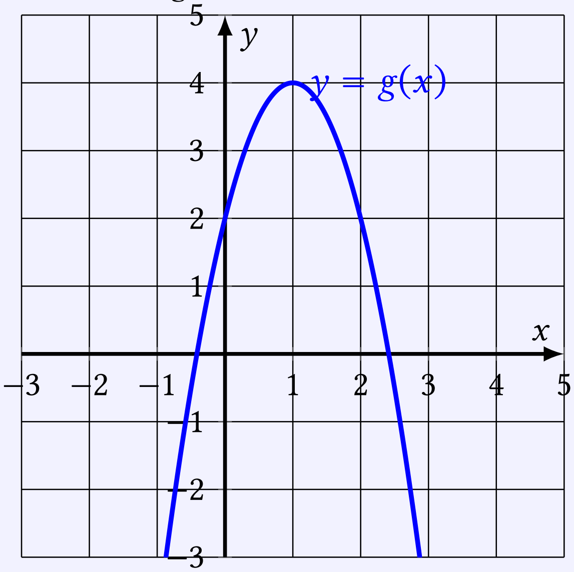
(1) $f(x) = x^2 - 1$ (2) $g(x) = |x - 1|$ (3) $h(x) = x^3 - 2x$ (4) $k(x) = \dfrac{1}{x^2}$.

Definition 1.5.5 Let c be a positive number. The function $g(x) = cf(x)$ is called a **vertical stretch** or **vertical compression** of $y = f(x)$ by a factor of c if $c > 1$ or $0 < c < 1$ respectively.

Remark If $a < 0$, then $g(x) = cf(x)$ is a combination of a vertical stretch or compression with a vertical reflection.

Example 1.5.8 The point $(9, -15)$ is on the graph of $y = f(x)$. Find a point on the graph of $g(x) = \dfrac{1}{3}f(x)$.

Example 1.5.9 The function $y = g(x)$ given in the following graph can be obtained from $f(x) = x^2$ by a combination of shifting, reflecting, and stretching. Describe the transformation and find an equation of g.



Definition 1.5.6 Let c be a positive number. The function $g(x) = f(cx)$ is called a **horizontal stretch** or **horizontal compression** of $y = f(x)$ by a factor of $\frac{1}{c}$ if $0 < c < 1$ or $c > 1$ respectively.

Remark If $c < 0$, then $g(x) = f(cx)$ is a combination of a horizontal stretch or compression with a horizontal reflection.

Example 1.5.10 The function $y = f(x)$ has two x-intercepts $(-2, 0)$ and $(4, 0)$. Determine if the function $g(x) = f(2x)$ has any x-intercepts. If so, find them. Otherwise explain why it has no x-intercept.

Example 1.5.11 Describe how to get the graph of the function $g(x) = 4x^2$ from the graph of the function $f(x)$.

How-to *The graph of the function $g(x) = Af(Bx + C) + D$ can be obtained by the following transformations in the given order.*

(1) *A vertical stretch/compression with the factor $|A|$ followed by a refection about x-axis if $A < 0$.*

(2) *A vertical shift of D units*

(3) *A horizontal shift of C units.*

(4) *A horizontal stretch/compression with the factor $\frac{1}{|B|}$ followed by a refection about y-axis if $B < 0$.*

Remark Note the horizontal and vertical transformation may be switched.

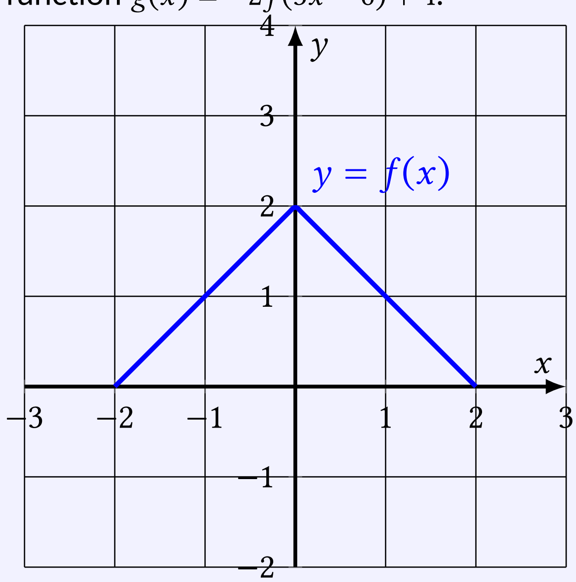
The order of horizontal or vertical transformation depends on how to get the point (x, y) from a point (a, b) on the original function under the substitutions $a = Bx + C$ and $y = Ab + D$.

To get x, one may add $-C$ to both sides first which corresponds to a horizontal shift of $-C$ units, and then multiply by $\frac{1}{|B|}$ which corresponds to a horizontal stretch/compression by a factor of $\frac{1}{B}$ (if B is negative, a horizontal reflection should be added). To get y, one may first multiply b by $|A|$ which corresponds to a vertical stretch/compression by a factor A (again if A is negative, a vertical reflection should be added) and then add D which corresponds to a vertical shift of D units. The scaling (dilation) is a stretch (or compression) if the multiplicative factor is greater than 1 (or less than 1 respectively).

Note one may also solve x from $a = Bx + C$ by multiplying $\frac{1}{B}$ first then add $-\frac{C}{B}$ which corresponds to horizontal stretch/compression by a factor $\frac{1}{B}$ followed by a horizontal shift by $-\frac{C}{B}$ units.

Similarly, one may also get y as $y = A(b + \frac{D}{A})$ which leads to a vertical shift of $\frac{D}{A}$ units followed by a vertical stretch/compression by a factor A.

Example 1.5.12 Using the graph of the function $y = f(x)$ given below to sketch the graph of the function $g(x) = -2f(3x - 6) + 4$.



Example 1.5.13 Sketch the graph of the function $g(x) = 2\sqrt{3x-1} - 4$ by a sequence of transformation applied on the graph of $f(x) = \sqrt{x}$.

Example 1.5.14 Find an equation of the function $y = g(x)$ whose graph is obtained from $f(x) = \sqrt{x}$ by the following transformations in the given order.

(1) Stretch vertically by a factor of 2

(2) Shift downward 2 units

(3) Shift 3 units to the left

(4) Stretch horizontally by a factor $\frac{1}{2}$.

Exercises

✍ **Exercise 1.5.1** Consider the functions $f(x) = x^2$, $g(x) = (x+1)^2 - 2$ and $h(x) = (x-2)^2 + 1$.

(1) Describe how to get the graph of g from the graph of f.

(2) Describe how to get the graph of h from the graph of g.

✍ **Exercise 1.5.2** Determine if the function is even, odd, or neither.

(1) $f(x) = 1 - x^2$. (2) $g(x) = \sqrt[3]{-x}$. (3) $g(x) = x^4 - x^3$.

✍ **Exercise 1.5.3** Sketch the graph of the function $g(x) = 2|3x - 6| + 4$ by a sequence of transformation applied on the graph of $f(x) = |x|$.

✍ **Exercise 1.5.4** Find an equation of the function $y = g(x)$ whose graph is obtained from $f(x) = \sqrt[3]{x}$ by the following transformations in the given order.

(1) Compress vertically by a factor of $\frac{1}{2}$.

(2) Reflect vertically.

(3) Shift downward 2 units.

(4) Compress horizontal by a factor 2.

(5) Shift 3 units to the right.

1.6 Inverse Functions

Definition 1.6.1 Let $y = f(x)$ be a one-to-one function with the domain A. A function $f^{-1}(x)$ is an **inverse function** of f if $f^{-1}(f(x)) = x$ for all x in A.

The notation f^{-1} is read "f inverse."

Remark

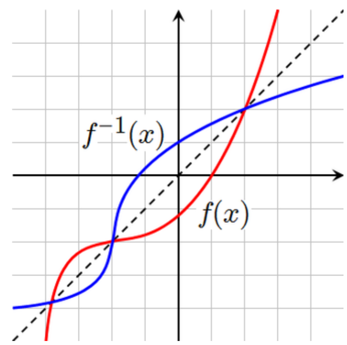
(1) If f is a one-to-one function, then it has a unique inverse function f^{-1}. Here is the proof. Suppose g is also an inverse f. Then $f(g(x)) = x = f(f^{-1}(x))$. Then $g(x) = f^{-1}(f(g(x))) = f^{-1}(f(f^{-1}(x))) = f^{-1}(x)$.

(2) Note that if f^{-1} is the inverse of f, then f is also the inverse of f^{-1} that is $f(f^{-1})(x) = x$ for all x in the domain of f^{-1}.

(3) In general, $f^{-1}(x) \neq f(x)^{-1}$.

(4) The graphs of a one-to-one function f and its inverse f^{-1} are symmetric about the diagonal line $y = x$.

(5) Suppose f has the domain A and the range B, then f^{-1} has the domain B and the range B (and vice verse).



The above graph of f and f^{-1} is taken from Wikipedia.

Example 1.6.1 Let f be a one-to-one function with $f(3) = 4$ and $f(4) = 5$. Find $f^{-1}(4)$.

Example 1.6.2 Let $f(x) = \dfrac{1}{x-1}$ and $g(x) = \dfrac{x+1}{x}$. Determine if g is the inverse function of f.

Example 1.6.3 Consider the function $f(x) = x^2 + 1$ with $x > 0$. Sketch the graph of $y = f^{-1}(x)$ without finding its equation.

How-to *Given a function $y = f(x)$, the inverse function is the solution y of the equation $f(y) = x$. The domain and the range of f and f^{-1} can be obtained from the domains of f and f^{-1}.*

Example 1.6.4 Consider the function $f(x) = 2x - 3$. Find the inverse function f^{-1} and its domain and range.

Example 1.6.5 Consider the function $f(x) = \dfrac{x}{x-1}$. Find the inverse function f^{-1} and its domain and range.

Example 1.6.6 Consider the function $f(x) = 2(x+1)^3 - 1$. Find the inverse function f^{-1} and its domain and range.

Example 1.6.7 Consider the function $f(x) = \sqrt{x-2}$. Find the inverse function f^{-1} and its domain and range.

Example 1.6.8 Find the inverse of each of the following functions if it exists.

Constant	Identity	Quadratic	Cubic	Reciprocal
$f(x) = c$	$f(x) = x$	$f(x) = x^2$	$f(x) = x^3$	$f(x) = \frac{1}{x}$
Reciprocal squared	Cube Root	Square Root	Absolute Value	
$f(x) = \frac{1}{x^2}$	$f(x) = \sqrt[3]{x}$	$f(x) = \sqrt{x}$	$f(x) = \lvert x \rvert$	

Exercises

✍ **Exercise 1.6.1** Let f be a one-to-one function with $f(-2) = -3$ and $f(-3) = 4$. Find $f^{-1}(-3)$.

✍ **Exercise 1.6.2** Let $f(x) = x^3 - 1$ and $g(x) = \sqrt[3]{x+1}$. Is $g = f^{-1}$?

✍ **Exercise 1.6.3** Consider the function $f(x) = \dfrac{1}{x-1} + 1$. Sketch the graph of f^{-1} without finding its equation.

✍ **Exercise 1.6.4** Consider the function $f(x) = \dfrac{1-x}{x+1}$. Find the inverse function f^{-1} and its domain and range.

✍ **Exercise 1.6.5** Consider the function $f(x) = 3(x-1)^3 + 2$. Find the inverse function f^{-1} and its domain and range.

✍ **Exercise 1.6.6** Consider the function $f(x) = \sqrt{x+1} - 1$. Find the inverse function f^{-1} and its domain and range.

Topic 2. Polynomial and Rational Functions

2.1 Quadratic Functions

Definition 2.1.1 A function $f(x) = ax^2 + bx + c$ with $a \neq 0$ is called a **quadratic function**. Its graph is called a **parabola**. By completing the square (let $h = -\dfrac{b}{2a}$ and $k = f(h)$), a quadratic function can be written in the **standard form** (or **vertex form**): $f(x) = a(x-h)^2 + k$. The vertical line $x = -\dfrac{b}{2a}$ (or $x = h$) is called the **axis of symmetry**. The **vertex** (h, k) is the intersection of the axis of symmetry and the parabola.

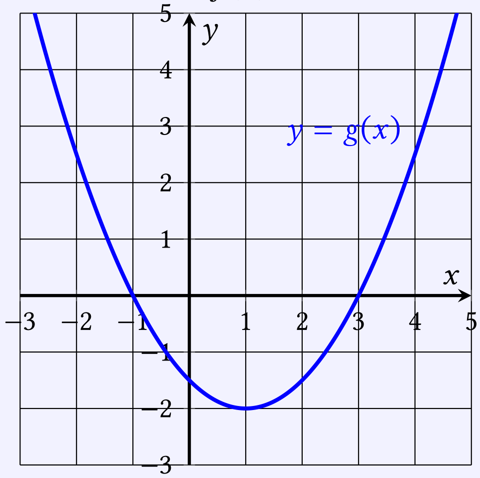
Note *The y-intercept of a quadratic function is $(0, f(0))$. The x-coordinates of x-intercepts are the zeros (or roots) of the function f, that is, the solutions of the equation $f(x) = 0$.*

Example 2.1.1 Find the vertex form of the quadratic function $f(x) = 2x^2 + 4x + 1$ and determine the vertex, axis of symmetry, x-intercepts, and y-intercept of the function.

Note

- *A quadratic function $f(x) = ax^2 + bx + c$ can be obtained from $y = x^2$ by a combination of vertical stretch by a factor $|a|$, a vertical reflection if $a < 0$, a vertical shift of $f\left(-\frac{b}{2a}\right)$ units, and a horizontal shift of $-\frac{b}{2a}$ units.*
- *The domain of a quadratic function is $(-\infty, \infty)$.*
- *If $a > 0$, then the parabola opens upward, the function has an absolute minimum $f\left(-\frac{b}{2a}\right)$, and the domain of the function is $\left[f\left(-\frac{b}{2a}\right), \infty\right)$.*
- *If $a < 0$, then the parabola opens downward, the function has an absolute maximum $f\left(-\frac{b}{2a}\right)$, and the domain of the function is $\left(-\infty, f\left(-\frac{b}{2a}\right)\right]$.*

Example 2.1.2 Find the vertex form equation for the quadratic function g in figure below as a transformation of $f(x) = x^2$, and then simplify the equation into general form.



Example 2.1.3 Find the domain and range of each function.

(1) $f(x) = 3x^2 + 6x - 5.$ (2) $f(x) = -2x^2 + 4 - 1.$

Example 2.1.4 A backyard farmer wants to enclose a rectangular space for a new garden within her fenced backyard. She has purchased 80 feet of wire fencing to enclose three sides, and she will use a section of the backyard fence as the fourth side. What's the maximal possible area of the garden.

Example 2.1.5 A local newspaper currently has 84,000 subscribers at a quarterly charge of \$30. Market research has suggested that if the owners raise the price to \$32, they would lose 5,000 subscribers. Assuming that subscriptions are linearly related to the price, what price should the newspaper charge for a quarterly subscription to maximize their revenue?

Example 2.1.6 A ball is thrown upward from the top of a 40-foot-high building at a speed of 80 feet per second. The ball's height above ground can be modeled by the equation $H(t) = -16t^2 + 80t + 40$.

(1) When does the ball reach the maximum height?

(2) What is the maximum height of the ball?

(3) When does the ball hit the ground?

Exercise

✍ **Exercise 2.1.1** For each of the following functions, (a) $f(x) = x^2 - 4x + 1$,(b) $f(x) = -2x^2 - 4x + 1$,

(1) write the function in vertex form,

(2) find the axis of symmetry,

(3) find the vertex,

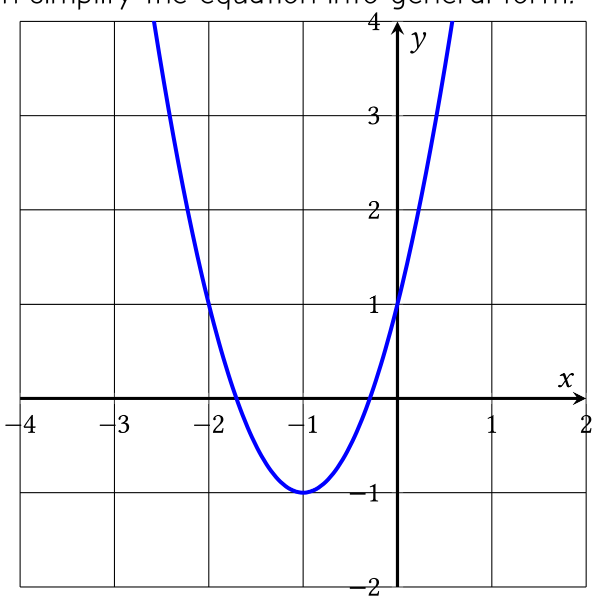
(4) find the y-intercept,

(5) find the x-intercepts if they exist,

(6) find the domain and range,

(7) find the global maximum or minimum if it exists.

✍ **Exercise 2.1.2** Find the vertex form equation for the quadratic function f in figure below, and then simplify the equation into general form.



✍ **Exercise 2.1.3** Find the dimensions of the rectangular parking lots producing the greatest area given that 500 feet of fencing will be used to for three sides.

✍ **Exercise 2.1.4** A rocket is launched in the air. Its height, in meters above sea level, as a function of time, in seconds, is given by $h(t) = -4.9t^2 + 229t + 234$. Find the maximum height the rocket attains.

✍ **Exercise 2.1.5** A soccer stadium holds $62,000$ spectators. With a ticket price of $\$11$, the average attendance has been $26,000$. When the price dropped to $\$9$, the average attendance rose to $31,000$. Assuming that attendance is linearly related to ticket price, what ticket price would maximize revenue?

2.2 Power and Polynomial Functions

Definition 2.2.1 A **power function** is a function that can be represented in the form

$$f(x) = kx^p,$$

where k and p are real numbers, and k is known as the **coefficient**.

Example 2.2.1 Determine if the function is a power function.

(1) $f(x) = -2x^3$ (2) $f(x) = \frac{1}{x^2}$ (3) $f(x) = \sqrt[3]{x}$ (4) $f(x) = 2^x$ (5) $f(x) = 2x^2 \cdot 3x^5$ (6) $f(x) = \frac{x}{x+1}$

Definition 2.2.2 The **end behavior** of a function f is the general direction that the function f approaches as x goes to ∞ or $-\infty$.

We use an arrow $\to$ to describe "goes to" or "approaches to". The notation $x \to \infty$ or $x \to -\infty$ means "x goes to infinity" or "x goes to negative infinity" respectively. The notation $f(x) \to \infty$ or $f(x) \to -\infty$ means "$f(x)$ goes to infinity" or "$f(x)$ goes to negative infinity" respectively.

If $f(x) \to b$ as $x \to \infty$ or $x \to -\infty$, then we say the line $y = b$ is a **horizontal asymptote**.

How-to *To determine the end behavior of a function f, take a large positive number N.*

If $f(N)$ is a large positive number, then $f(x) \to \infty$ as $x \to \infty$.

If $-f(N)$ is a large positive number, then $f(x) \to -\infty$ as $x \to \infty$.

If $f(-N)$ is a large positive number, then $f(x) \to \infty$ as $x \to -\infty$.

If $-f(-N)$ is a large positive number, then $f(x) \to -\infty$ as $x \to -\infty$.

Example 2.2.2 Determine the end behavior(s) of the function.

(1) $f(x) = -2x^3$ (2) $f(x) = \frac{1}{x^2}$ (3) $f(x) = \sqrt[3]{x}$

Definition 2.2.3 Let n be a non-negative integer. A **polynomial function** of **degree** n is a function that can be written in the form

$$f(x) = a_n x^n + \cdots + a_2 x^2 + a_1 x + a_0.$$

- Each a_i is called a **coefficient**.
- Each product $a_i x^i$ is called a **term** of a polynomial function.
- The term $a_n x^n$ is called the **leading term**. The number a_n is called the **leading coefficient**.
- The number a_0 is called the **constant term**.

Note *The end behavior of a polynomial function $f(x) = a_n x^2 + \cdots + a_0$ of degree n is completely determined by the end behavior of the power function $g(x) = a_n x^n$.*

The domain of a polynomial function is $(-\infty, \infty)$. The range of an odd degree polynomial function is also $(-\infty, \infty)$. The range of an even degree polynomial function is either $[y_{min}, \infty)$ if $a_n > 0$ or $(-\infty, y_{max}]$ if $a_n < 0$, where y_{min} (respectively, y_{max}) is the absolute minimum (respectively, maximum) of the function.

Example 2.2.3 Determine the end behavior of the function.

(1) $f(x) = 2x^4 - 3x + 1$ (2) $g(x) = -3x^3 + 2x^2 - x$ (3) $h(x) = -4x^6 - 7x^5 + 10x^4 + 2$

Example 2.2.4 Identify the degree, the leading therm and the end behavior of the polynomial function.

(1) $f(x) = -3x^2(x-1)(x+4)$ (2) $f(x) = 2x^3(1-x)(x+1)$

Definition 2.2.4 If f is a polynomial function, then a number c is called a **zero** of f if $f(c) = 0$.

Proposition 2.2.5 *Let f be a polynomial and c a real number. Then the following are equivalent:*

(1) *c is a zero of f.*

(2) *$x = c$ is a solution of the equation $f(x) = 0$.*

(3) *$x - c$ is a factor of $f(x)$.*

(4) *$(c, 0)$ is an x-intercept of the function of $y = f(x)$.*

Example 2.2.5 Find x-intercepts and the y-intercept of the polynomial function $f(x) = x^3 + 3x^2 - x - 3$.

Example 2.2.6 Find x-intercepts and the y-intercept of the polynomial function $f(x) = x^4 + 2x^2 - 3$.

Definition 2.2.6 A **turning point** (also known as a local extremum) is a point at which the function values change from increasing to decreasing or decreasing to increasing.

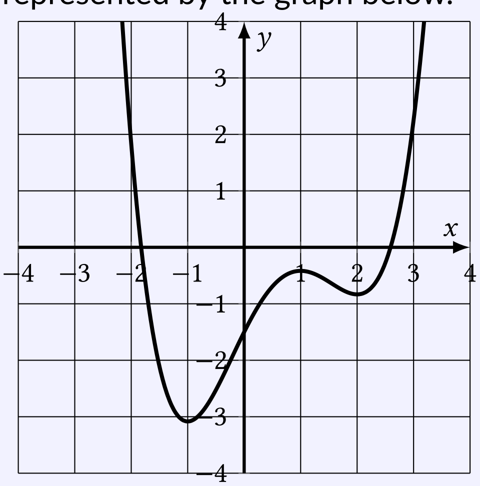
Theorem 2.2.7 (Fundamental Theorem of Algebra[1]) *A degree n polynomial function has at least one complex zero.*

Proposition 2.2.8 *A degree n polynomial function may have at most n real zeros and $n - 1$ turning points.*

[1] A relatively elementary proof can be found at https://tinyurl.com/tFToA.

Example 2.2.7 Consider the polynomial function $f(x) = (x-2)(x+1)(x-4)$. Determine the zeros, the number of turning points, the x-intercepts, and the y-intercept.

Example 2.2.8 What can we conclude about the leading term of the polynomial function $y = f(x)$ represented by the graph below.



Exercises

✍ **Exercise 2.2.1** Find the degree and leading coefficient, and determined the end behavior for the given polynomial.
(1) $f(x) = -2x^4$(2) $f(x) = 2x^5 - x^3$(3) $f(x) = -2x(1 - x^2)$(4) $f(x) = (x^2 - 1)(2x - 1)(x + 2)$

✍ **Exercise 2.2.2** Find x-intercepts (if they exist) and the y-intercept of the polynomial function.
(1) $f(x) = -2x^4 + x^2 + 1$ (2) $f(x) = 2x + x^3 - 3x^5$ (3) $f(x) = x^3 + x^2 - 4x - 4$

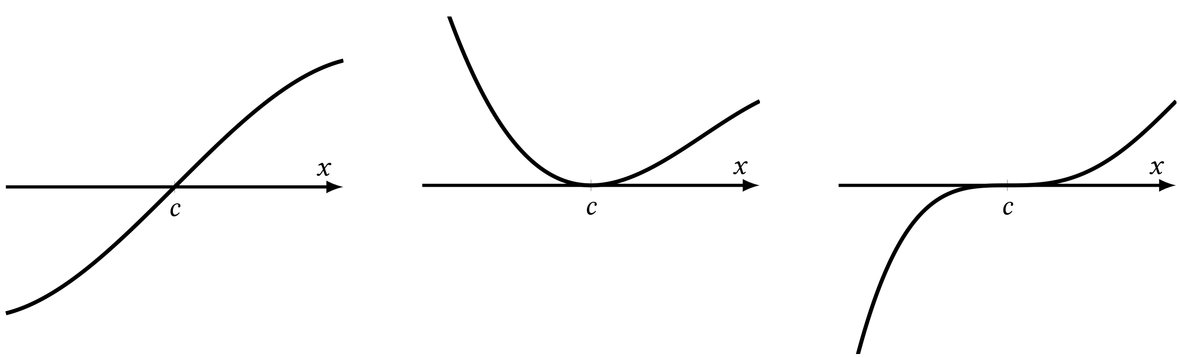
2.3 Graphs of Polynomial Functions

Definition 2.3.1 We say a zero c of a polynomial function f has the **multiplicity** k if $f(x) = (x - c)^k g(x)$ and c is not a zero of g.

Example 2.3.1 Find the zeros of the polynomial function $f(x) = x^3(x-1)^2(x-2)$ and determine their multiplicities.

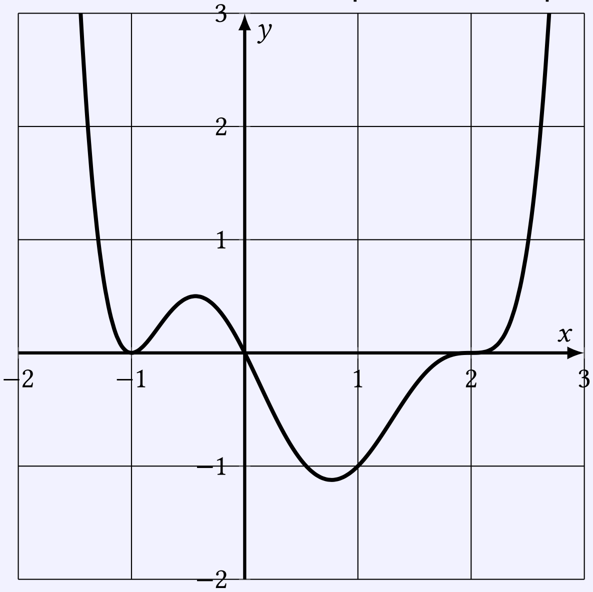
Example 2.3.2 A polynomial function P of degree 3 has two zeros 1 and 2 with multiplicity 2 and 1 respectively. The y-intercept is $(0, -4)$. Find an equation for P.

Note (Local Graph Near a Zero) *Let f be a polynomial with positive leading coefficient and c is a zero of f of the multiplicity m. The local shape of a polynomial function with positive leading coefficient near a zero is of one of the following types, or a vertical reflection of that.*



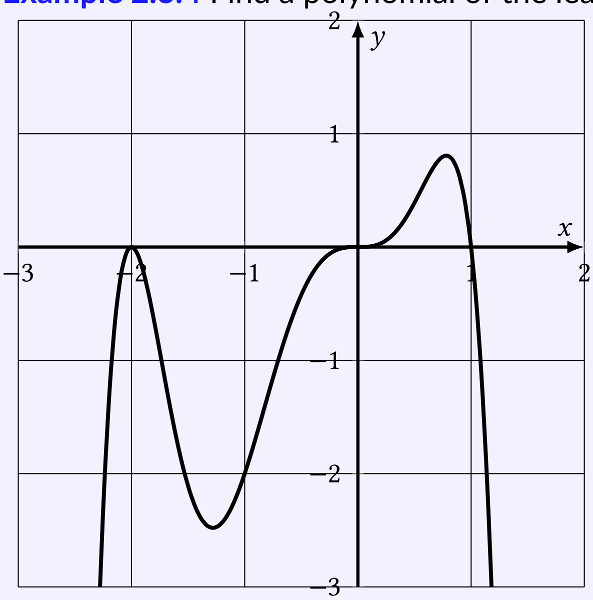
$k = 1$ | $k > 1$ *and* k *is even* | $k > 1$ *and* k *is odd*

Example 2.3.3 Use the graph of the function of degree 6 in the figure below to identify the zeros of the function and their possible multiplicities.



A polynomial of degree 6.

Example 2.3.4 Find a polynomial of the least degree whose graph is given below.



Definition 2.3.2 A **continuous** function has no breaks in its graph. A **smooth** function is a continuous function whose graph that has no sharp corners.

Note *Polynomial functions are smooth functions.*

Theorem 2.3.3 (Intermediate Value Theorem[2]) *If f is a continuous function and $f(a)f(b) < 0$, then there exists at least one value c between a and b such that $f(c) = 0$. In particular, the theorem holds true for polynomial functions.*

Corollary 2.3.4 *Let f be a polynomial function, a and b real zeros of f. If f has no other zeros between a and b, then either $f(x) > 0$ for all x between a and b or $f(x) < 0$ for all x between a and b.*

Theorem 2.3.5 (Rolle's Theorem for Polynomial Functions) *Let f be a polynomial function, a and b two zeros. Then f has at lease one local extremum (turning point) between a and b.*

Example 2.3.5 Determine if the polynomial function $f(x) = 5x^4 - 2x^3 - 20$ has a zero on the interval $[1, 2]$.

How-to (Graph a Polynomial Function)

(1) *Plot the y-intercept.*

(2) *Determine the real zeros and their multiplicities, and sketch local graph near x-intercepts.*

(3) *Determine the end behavior and sketch the graph of the left and right tails.*

(4) *Using symmetry to plot additional points if possible.*

(5) *Use test points to determine whether the graph of the polynomial lies above or below the x-axis over the intervals between zeros, and estimate the locations of turning points.*

(6) *Connect points and local graphs smoothly.*

[2] A proof of the theorem can be found in https://tinyurl.com/ivtcont.

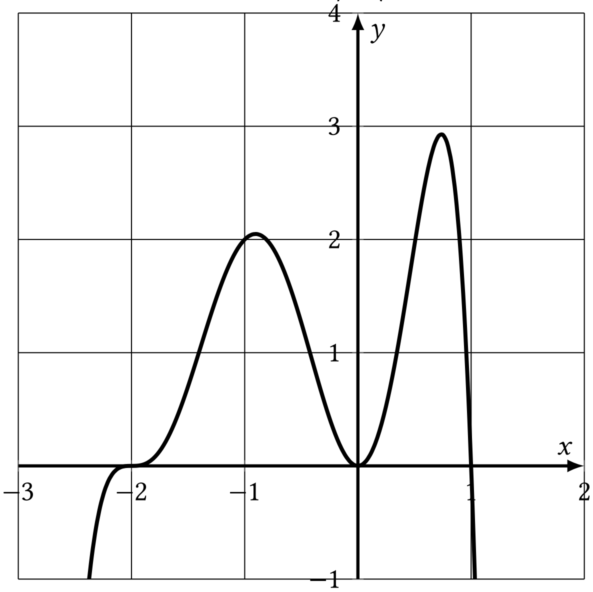
Example 2.3.6 Sketch the graph of the polynomial function $f(x) = (x-4)(x-1)^2(x+3)$.

Exercises

✍ **Exercise 2.3.1** Find the t-intercepts and the P-intercept of the polynomial function $P(t) = 3t^4 - 15t^3 + 12t^2$.

✍ **Exercise 2.3.2** A polynomial function P of degree 3 has two zeros 1 and 2 with multiplicity 2 and 1 respectively. The y-intercept is $(0, -4)$. Find an equation for P.

Exercise 2.3.3 Find a polynomial of the least degree whose graph is given below.



Exercise 2.3.4 Sketch the graph of the polynomial function $f(x) = x^4 - 2x^3 + x^2$.

2.4 Dividing of Polynomials

Theorem 2.4.1 (Division Algorithm) *Let $p(x)$ and $d(x)$ be two polynomial. Suppose that $d(x)$ is non-zero and the degree of $d(x)$ is less than or equal to the degree of $f(x)$. Then there exist unique polynomials $q(x)$ and $r(x)$ such that*

$$p(x) = d(x)q(x) + r(x)$$

and the degree of $r(x)$ is less than the degree of $d(x)$.

Definition 2.4.2 In the above theorem, $p(x)$ is called the **dividend**, $d(x)$ is called the **divisor**, $q(x)$ is called the **quotient** and $r(x)$ is called the **remainder**. If $r(x) = 0$, then we say that $d(x)$ **divides** $p(x)$.

A divison algorithm[3] is an algorithm which computes the quotient and the reminder.

Polynomial long division is a divison algorithm. Another shorthand division algorithm is the synthetic division.

Example 2.4.1 Divide $6x^3 + 11x^2 - 31x + 15$ by $3x - 2$.

Example 2.4.2 Divide $4x^4 + 3x^2 - 1x + 5$ by $2x^2 - x + 3$.

[3] See wikipedia page on Polynomial long division for various division algorithms.

Definition 2.4.3 Synthetic division is a shortcut that can be used when the divisor is linear binomial in the form $x - c$. In synthetic division, only the coefficients are used in the division process.

Example 2.4.3 Use synthetic division to divide $4x^3 + 10x^2 - 6x - 20$ by $x + 2$.

Example 2.4.4 Use synthetic division to divide $-9x^4 + 10x^3 + 7x^2 - 6$ by $x - 1$.

Exercises

✍ **Exercise 2.4.1** Divide $3x^2 - 7x - 3$ by $3x - 1$.

✍ **Exercise 2.4.2** Divide $16x^3 - 12x^2 + 20x - 3$ by $4x + 5$.

✍ **Exercise 2.4.3** Use synthetic division to divide $5x^2 - 3x - 36$ by $x - 3$.

✍ **Exercise 2.4.4** Divide $2x^4 + 4x^3 - 3x^2 - 5x - 2$ by $x + 2$.

2.5 Zeros of Polynomials

Theorem 2.5.1 (The Remainder Theorem) *If a polynomial $f(x)$ is divided by $x-c$, then the remainder is the value $f(c)$.*

Example 2.5.1 Use the Remainder Theorem to evaluate $f(x) = 6x^4 - x^3 - 15x^2 + 2x - 7$ at $x = 2$.

Theorem 2.5.2 (The Rational Zero Theorem) *Let $f(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0$ be polynomial with integer coefficients. Then every rational zero of $f(x)$ is in the form $\frac{p}{q}$, where p is a factor of the constant term a_0 and q is a factor of the leading coefficient a_n.*

Example 2.5.2 List all possible rational zeros of $f(x) = 2x^4 - 5x^3 + x^2 - 4$.

Example 2.5.3 Find the zeros of $f(x) = 4x^3 - 3x - 1$.

Theorem 2.5.3 (Linear Factorization Theorem) *Let $f(x)$ be a polynomial with the degree $n > 1$ and the leading coefficient a_n. Then*

$$f(x) = a_n(x - c_1)(x - c_2)\cdots(x - c_n),$$

where c_i are complex numbers.

Theorem 2.5.4 (Complex Conjugation Theorem) *Let $f(x)$ be a polynomial. If $x - (a + b\,i)$ is a factor of f, then $x - (a - b\,i)$ is also a factor of f.*

Example 2.5.4 Find a fourth degree polynomial with real coefficients that has zeros of -3, 2, i, such that $f(-2) = 100$.

Theorem 2.5.5 (Descartes' Rule of Signs[4]) *Let $f(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0$ be a polynomial function with real coefficients.*

- *The number of positive real zeros counted with multiplicity is either equal to the number of sign changes of $f(x)$ or is less than the number of sign changes by an even integer.*
- *The number of negative real zeros counted with multiplicity is either equal to the number of sign changes of $f(-x)$ or is less than the number of sign changes by an even integer.*

Example 2.5.5 Use Descartes' Rule of Signs to determine the possible numbers of positive and negative real zeros for $f(x) = -x^4 - 3x^3 + 6x^2 - 4x - 12$.

[4] For a proof, see the blogpost Proof of Descartes' Rule of Signs

Exercises

✍ **Exercise 2.5.1** Find all zeros of $f(x) = 2x^3 + 5x^2 - 11x + 4$.

✍ **Exercise 2.5.2** Find all zeros of $f(x) = x^4 + 3x^3 + 2x^2 - 2x - 4$.

✍ **Exercise 2.5.3** Find a fourth degree polynomial with real coefficients that has zeros of -1, 2, $1 + i$, such that $f(-2) = 10$.

2.6 Rational Functions

Definition 2.6.1 Let $p(x)$ and $q(x)$ be polynomials with $\deg(q(x)) > 0$. The function $f(x) = \dfrac{p(x)}{q(x)}$ is called a rational function. The domain of f is $\{x \mid q(x) \neq 0\}$.

Example 2.6.1 Find the domain of $f(x) = \dfrac{x+3}{x^2-9}$.

Definition 2.6.2 (Vertical Asymptote) A **vertical asymptote** of a function f is a vertical line $x = a$ where the graph of f goes to positive or negative infinity as x approached a from left or right, that is, as $x \to a^-$ or a^+, $f(x) \to \infty$, or as $x \to a^-$ or a^+, $f(x) \to -\infty$, where $x \to a^-$ (or a^+) means x approaches a from the left (respectively, right).

We say a function f has a **removable discontinuities** (or **hole**) at $x = a$ if $f(x) \to b$ as $x \to a$ but $f(a)$ is undefined.

Proposition 2.6.3 *Let $f = \dfrac{p(x)}{q(x)}$ be a rational function. If $p(a) = q(a) = 0$, then f has a hole at a. If $q(a) = 0$ but $p(a) \neq 0$, then f has a vertical asymptote $x = a$.*

Definition 2.6.4 (Horizontal Asymptote) A **horizontal asymptote** of a function f is a horizontal line $y = b$ where the graph of f approaches to b as x goes to positive or negative infinity, that is, as $x \to \infty$, or $x \to -\infty$, $f(x) \to b$.

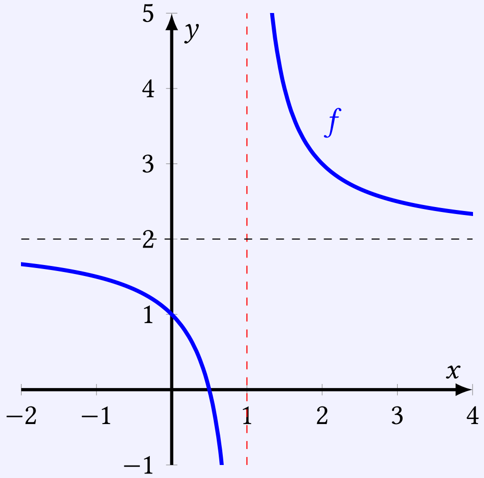
Definition 2.6.5 (Slanted Asysmptote) A **slanted asymptote** of a function f is a line $y = mx + b$ with $m \neq 0$ where the graph of f approaches to $mx + b$ as x goes to positive or negative infinity, that is, as $x \to \infty$ or $x \to -\infty$, $f(x) \to mx + b$.

Note *For a rational function f, as $x \to \infty$ or $-\infty$, $f(x)$ approaches the asymptote only from one side of the line. This information will be helpful when sketching a graph.*

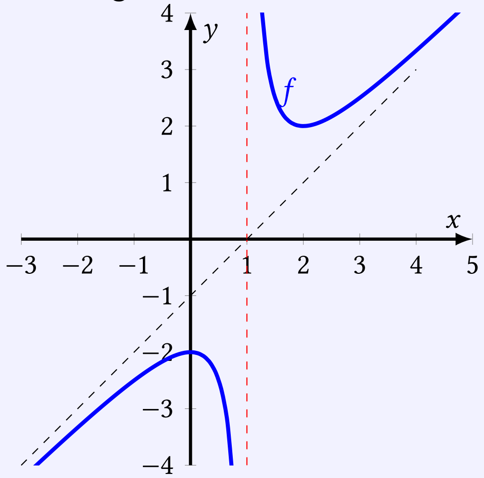
Proposition 2.6.6 *Let $f(x) = \dfrac{p(x)}{q(x)} = \dfrac{a_m x^m + a_{m-1}x^{m-1} + \ldots + a_1 x + a_0}{b_n x^n + b_{n-1}x^{n-1} + \ldots + b_1 x + b_0}$ be a rational function.*

- *if $m < n$, then f has a horizontal asymptote $x = 0$;*
- *if $m = n$, then f has a horizontal asymptote $x = \frac{a_m}{b_n}$;*
- *if $m = n + 1$, then f has a slated asysmptote $y = mx + b$, where $mx + b$ is the quotient of $\frac{p(x)}{q(x)}$.*
- *if $m > n + 1$, then f has no horizontal or slated asymptote;*

Example 2.6.2 Use arrow notation to describe asymptotes of the function f graphed in the figure.



Example 2.6.3 Use arrow notation to describe the slanted asymptote of the function f graphed in the figure.



Example 2.6.4 Find the asymptotes of the rational function $f(x) = \dfrac{x^2+1}{2x^2-3x+1}$ if they exist.

Example 2.6.5 Find the asymptotes of the rational function $f(x) = \dfrac{-x^2 + 3x - 1}{x - 1}$ if they exist.

Example 2.6.6 Find the asymptotes and holes of the function $f(x) = \dfrac{x^2 + x - 6}{x^3 - 2x^2 - x + 2}$ if they exist.

How-to (Graph of a Rational Function)

(1) *Find the y-intercept and plot it.*

(2) *Find the x-intercept(s) and plot them.*

(3) *Find all vertical asymptotes and graph them as dashed lines.*

(4) *Find the horizontal asymptote or the slant asymptote (or neither), and graph the asymptote as a dashed line.*

(5) *Plot a test point in each interval whose boundary values are zeros of the denominators or the function to determine if the graph is above the axis or below the axis over that interval.*

(6) *Sketch the function based on the information found above.*

Example 2.6.7 Sketch a graph of $f(x) = \dfrac{(x+2)(x-3)}{(x+1)^2(x-2)}$.

Exercises

Example 2.6.8 Find asymptotes of the rational function $f(x) = \dfrac{3x^2 - 1}{x^2 + 4x - 5}$

✍ **Exercise 2.6.1** Find asymptotes of the rational function $f(x) = \dfrac{x^2}{x+1}$.

✍ **Exercise 2.6.2** Find asymptotes and holes of the rational function $f(x) = \dfrac{(x-1)(x-2)}{x^2 - 4}$.

✍ **Exercise 2.6.3** Sketch a graph of the rational function $f(x) = \dfrac{(x+2)^2(x-1)}{(x+1)^2(x-2)}$.

✍ **Exercise 2.6.4** Sketch a graph of the rational function $f(x) = \dfrac{4(x+2)(x-3)^3}{(x+1)(x-2)^2}$.

2.7 Polynomial and Rational Inequalities

How-to (Polynomial or Rational Inequalities)

(1) *Rewrite the inequality into the form $f(x)$ inequality symbol 0.*

(2) *Find all real zeros of $f(x)$.*

(3) *Break the* ***DOMAIN OF THE FUNCTION*** *(the whole number line if f is a polynomial) into intervals using zeros from the previous step.*

(4) *Choose a test point from each interval to determine the sign of f.*

(5) *Determine the solutions (union of intervals in which the test point satisfies the inequality) and whether the boundary values of the intervals should be included.*

Example 2.7.1 Solve the inequality $x^2 \leq 7x - 6$.

Example 2.7.2 Solve the inequality $2x^3 - 3x^2 > 3x - 2$.

Example 2.7.3 Solve the inequality $\dfrac{4-x}{x-1} < 2$.

Example 2.7.4 Solve the inequality $\dfrac{6x}{(x+1)(x+2)} \geq 1$.

Exercises

✍ **Exercise 2.7.1** Solve the inequality $-x^2 > 5x - 6$.

✍ **Exercise 2.7.2** Solve the inequality $2x^3 + x^2 \leq 2x + 1$.

✍ **Exercise 2.7.3** Solve the inequality $1 \geq \dfrac{x-1}{2x+1}$.

✍ **Exercise 2.7.4** Solve the inequality $\dfrac{x+8}{x^2-4} < 1$.

3.1 Exponential Functions

Definition 3.1.1 (Exponential Functions) For any real number x, an exponential function f is a function defined by an equation

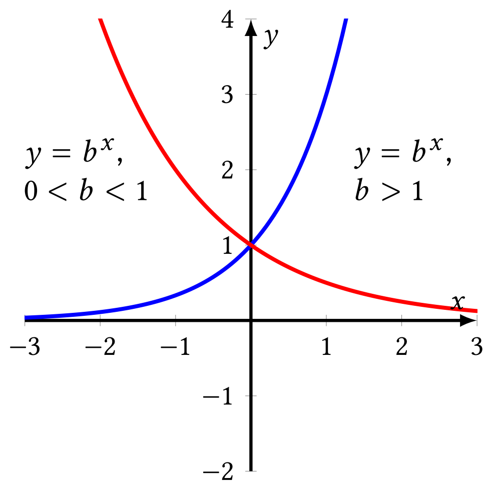
$$f(x) = b^x,$$

where b is a positive real number and $b \neq 1$.

Let $f(x) = b^x$ be an exponential function. Then

- the domain of f is $(-\infty, \infty)$,
- the range of f is $(0, \infty)$,
- the y-intercept is $(0, 1)$,
- f has a horizontal asymptote $y = 0$,
- f is increasing if $b > 1$,
- f is decreasing if $0 < b < 1$.

Note if $f(x) = ab^x$, then y-intercept is $(0, a)$ and $a = f(0)$ is known as the initial value.



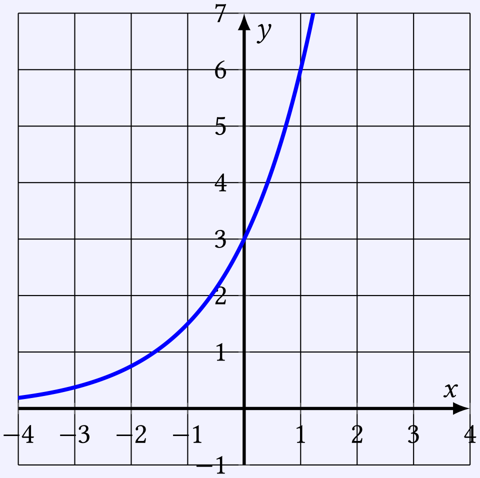
Example 3.1.1 The population of India was about 1.25 billion in the year 2013, with an annual growth rate of about 1.2%. This situation is represented by the growth function $P(t) = 1.25(1.012)^t$, where t is the number of years since 2013. To the nearest thousandth, what will the population of India be in 2031?

Example 3.1.2 In 2006, 80 deer were introduced into a wildlife refuge. By 2012, the population had grown to 180 deer. The population was growing exponentially. Write an algebraic function $N(t)$ representing the population N of deer over time t.

Example 3.1.3 Sketch the graph of the function $f(x) = 2 \cdot 3^{x+1} + 1$ by transforming the graph of the function $f(x) = 2 \cdot 3^x$.

Example 3.1.4 Find an exponential function $f(x) = ab^x$ that passes through the points $(-2, 6)$ and $(2, 1)$.

Example 3.1.5 Find an exponential function $f(x) = ab^x$ graphed in the following figure.



Definition 3.1.2 (The Natural Number e) The natrual number, denoted by e, the number that $\left(1+\frac{1}{n}\right)^n$ approaches to as n increases without bound. Approximately, $e \approx 2.718282$.

Example 3.1.6 Calculate $e^{3.14}$. Round to five decimal places.

Note (Compound Interest) *Let P be the initial amount of the account, known as the principal, r the annual interest rate, and t is the number of years. The balance A after t years is*

- $A(t) = P\left(1+\frac{r}{n}\right)^{nt}$ *if the interest is compounded n times per year.*
- $A(t) = Pe^{rt}$ *if the interest is compounded continuously ($n \to \infty$).*

Example 3.1.7 We invest $\$3,000$ in an investment account paying 3% interest compounded quarterly, how much will the account be worth in 10 years?

Example 3.1.8 A person invested $\$1,000$ in an account earning 10% per year compounded continuously. How much was in the account at the end of two and a half year?

Example 3.1.9 A 529 Plan is a college-savings plan that allows relatives to invest money to pay for a child's future college tuition; the account grows tax-free. Lily wants to set up a 529 account for her new granddaughter and wants the account to grow to $\$40,000$ over 18 years. She believes the account will earn 6% compounded semi-annually (twice a year). To the nearest dollar, how much will Lily need to invest in the account now?

Example 3.1.10 $Radon-222$ decays at a continuous rate of 17.3% per day. How much will $100mg$ of $Radon-222$ decay to in 3 days?

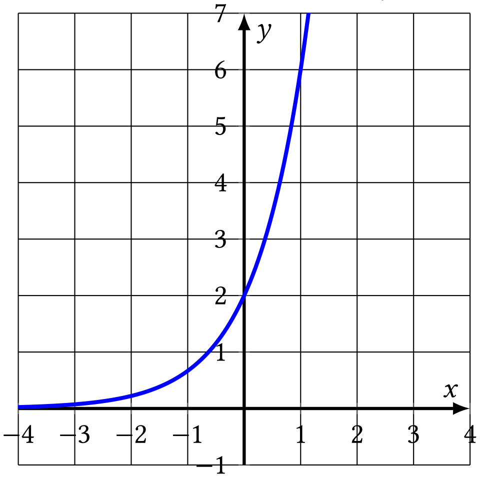
Exercises

✍ **Exercise 3.1.1** A vehicle depriciates according to the formula: $v = 27500\,(3.42)^{-.04x}$ where x is the age of the car in years. Find the value of the car when it is 14-years old.

✍ **Exercise 3.1.2** Sketch the graph of the function $f(x) = -2 \cdot 3^{x-2} + 2$.

✍ **Exercise 3.1.3** Find an exponential function $f(x) = ab^x$ that passes through the points $(-2, -6)$ and $(2, -1)$.

✍ **Exercise 3.1.4** Find an exponential function $f(x) = ab^x$ graphed in the following figure.



✍ **Exercise 3.1.5** A wolf population is growing exponentially. In 2011, 129 wolves were counted. By 2013, the population had reached 236 wolves. What two points can be used to derive an exponential equation modeling this situation? Write the equation representing the population N of wolves over time t.

✍ **Exercise 3.1.6** A scientist begins with 100 milligrams of a radioactive substance that decays exponentially. After 35 hours, 50 mg of the substance remains. How many milligrams will remain after 54 hours?

✍ **Exercise 3.1.7** Consider the function $f(x) = -\dfrac{1}{2e^{-x}+1}$. Find $f(0)$, $f(\sqrt{2})$ and $f(-1)$.

✍ **Exercise 3.1.8** An account is opened with an initial deposit of $\$6,500$ and earns 3.6% interest.

(1) What will the account be worth in 20 years if the interest is compounded monthly.

(2) What will the account be worth in 20 years if the interest is compounded continuously.

3.2 Logarithmic Functions

Definition 3.2.1 Let $y = b^x$ be an exponential function, where $b > 0$ and $b \neq 1$. Its inverse function is called the **logarithmic function with base** b, denoted as $\log_b$.

From the definition of inverse function, for any $x > 0$, $\log_b x$, read as the logarithm with base b of x, is the unique number such that

$$b^{\log_b x} = x.$$

In terms of equations,

$$y = \log_b x \quad \text{is equivalent to} \quad x = b^y.$$

Example 3.2.1 Write the following logarithmic equality in exponential form.

(1) $\log_6(\sqrt{6}) = \frac{1}{2}$ (2) $\log_3(9) = 2$ (3) $\log_2(x) = 3$ (4) $\log_x(5) = \frac{1}{3}$

Example 3.2.2 Use the exponential form to evaluate the logarithm.

(1) $\log_2 4$ (2) $\log_2 \sqrt{2}$ (3) $\log_9 3$

Definition 3.2.2 A **common logarithm** is a logarithm with base 10. We write $\log_{10}(x)$ simply as $\log(x)$.

A **natural logarithm** is a logarithm with base e, the natrual number. We write $\log_e(x)$ simply as $\ln(x)$.

Example 3.2.3 Evaluate the logarithm without using a calculator.

(1) $\log(1000)$ (2) $\ln(e^2)$

Example 3.2.4 Evaluate the logarithm using a calculator.

(1) $\log 2$ (2) $\ln 2$

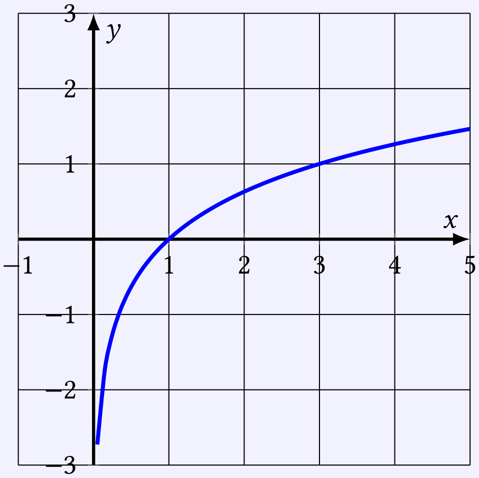
The domain of $y = \log_b x$ is $(0, \infty)$, and the range is $(-\infty, \infty)$. The function $\log_b$ has an x-intercept $(1, 0)$ and a vertical asymptote $x = 0$.

If $b > 1$, then $y = \log_b x$ is increasing. If $0 < b < 1$, then $y = \log_b x$ is decreasing.

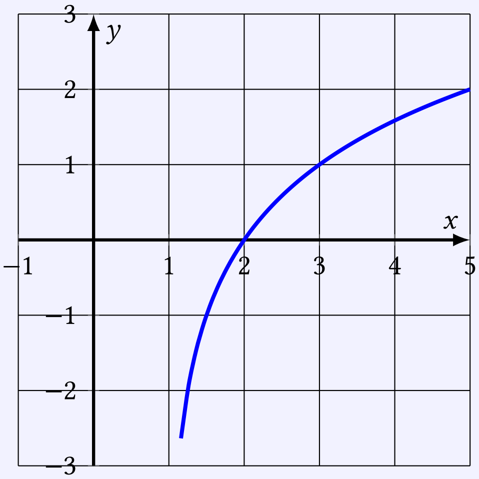
Example 3.2.5 Find the domain of the function.

(1) $f(x) = \log_2(x+3)$ (2) $f(x) = \log_3(3-2x)$ (3) $f(x) = \ln(4-x^2)$ (4) $f(x) = \log\left(\dfrac{x+1}{x-2}\right)$

Example 3.2.6 Find an equation for the function $y = \log_b x$ whose graph is shown below.



Example 3.2.7 Find an equation for the function $y = \log_b(x - a)$ whose graph is shown below.



Example 3.2.8 Find the vertical asymptote of $f(x) = -2\log_3(x + 4) + 5$

Exercises

✍ **Exercise 3.2.1** Write the following logarithmic equality in exponential form.

(1) $\log_4 2 = x$ (2) $\log_3(x) = 2$ (3) $\log_x(2) = \frac{1}{2}$

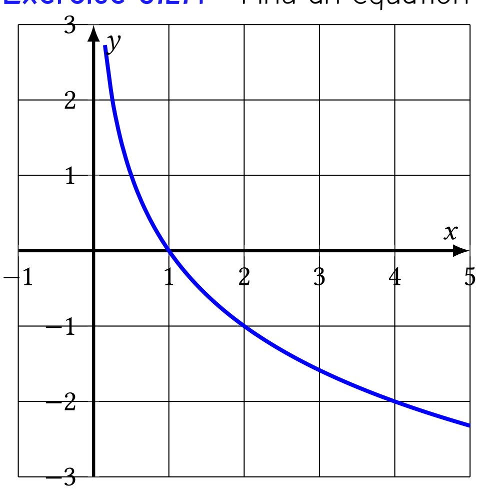
✍ **Exercise 3.2.2** Evaluate the logarithm using a calculator.

(1) $\log 3$ (2) $\ln 5$ (3) $\frac{\log 5}{\ln 3}$

✍ **Exercise 3.2.3** Find the domain of the function.

(1) $f(x) = \log_2(2x-1)$ (2) $f(x) = \ln(9-4x^2)$ (3) $f(x) = \log\left(\frac{1-x}{x-2}\right)$

✍ **Exercise 3.2.4** Find an equation for the function $y = -\log_b x$ whose graph is shown below.



✍ **Exercise 3.2.5** Find the vertical asymptote of $f(x) = -3\log_2(2x-1)+1$

3.3 Properties of Logarithms

Proposition 3.3.1 (Basic Properties of Logarithms) *Assume that* $b > 0$ *and* $b \neq 1$, *and* $x > 0$. *Then*

(1) $b^{\log_b x} = x$.

(2) $\log_b(b^x) = x$.

(3) $\log_b b = 1$ *and* $\log_b 1 = 0$.

Example 3.3.1 Evaluate the logarithm.

(1) $\log_8 2$ (2) $10^{\log(\frac{1}{2})}$ (3) $\log_3(e^0)$

Proposition 3.3.2 (Basic Properties of Logarithms) *Assume* $M > 0$, $N > 0$, $b > 0$ *and* $b \neq 1$. *Then*

(1) *(The product rule)* $\log_b(MN) = \log_b M + \log_b N$

(2) *(The quotient rule)* $\log_b(\frac{M}{N}) = \log_b M - \log_b N$.

(3) *(The power rule)* $\log_b(M^p) = p\log_b M$, *where* p *is any real number.*

(4) *(The change-of-base property)* $\log_b M = \dfrac{\log_a M}{\log_a b}$, *where* $a > 0$ *and* $a \neq 1$. *In particular,*

$$\log_b M = \frac{\log M}{\log b} \quad \text{and} \quad \log_b M = \frac{\ln M}{\ln b}.$$

Example 3.3.2 Expand the logarithmic expression.

(1) $\log_3(30x(3x+4))$ (2) $\log\left(\dfrac{2x^2+6x}{3x+9}\right)$ (3) $\log_2(\sqrt{x^2+1})$ (4) $\ln\left(\dfrac{x^4(y-1)}{x^2+1}\right)$

Example 3.3.3 Evaluate the logarithm using calculator.

(1) $\log_2 5$ (2) $\log_3 5 - \log 53$ (3) $\dfrac{\log_2 3}{\log_3 2}$

Example 3.3.4 Expand the logarithmic expression

$$\ln\left(\frac{\sqrt{(x-1)(2x+1)^2}}{(x^2-9)}\right).$$

Example 3.3.5 Condense the logarithmic expression.

(1) $\log_2(x^2) + \frac{1}{2}\log_2(x-1) - 3\log_2((x+3)^2)$ (2) $3\ln(x) - \frac{1}{2}\ln(x+1) - 2\ln(\sqrt{x^2+3})$

Exercises

✍ **Exercise 3.3.1** Expand the logarithmic expression.

(1) $\log_6\left(\dfrac{64x^3(4x+1)}{(2x-1)}\right)$.

(2) $\ln\left(\dfrac{\sqrt{(x-1)}(2x+1)^2}{(x^2-9)}\right)$

✍ **Exercise 3.3.2** Condense the logarithmic expressions.

(1) $2\log x - 4\log(x+5) + \dfrac{1}{3}\log(3x+5)$

(2) $4(3\log(x) + \log(x+5) - \log(2x+3))$

✍ **Exercise 3.3.3** Evaluate the logarithm using a calculator.

(1) $\log_5 23$

(2) $\log_4 9 - \log 5$

(3) $\dfrac{\ln 10}{\log_5 10}$

3.4 Exponential and Logarithmic Equations

How-to (Exponential equation) *Isolate an exponential expression first and then take logarithm with the same base, or any base of both sides. After that, solve the resulting equation.*

Example 3.4.1 Solve

(1) $3^{x+1} = 4$ (2) $2^{x-1} = 2^{2x-4}$ (3) $8^{x+2} = 16^{x+1}$ (4) $5^{x+2} = 4^{x}$

Example 3.4.2 Solve

(1) $100 = 20e^{2t}$ (2) $4e^{2x} + 5 = 12$ (3) $e^{2x} - e^{x} = 56$

How-to (Logarithmic equation) *Isolate a logarithmic expression first and then apply the exponential function with the same base. After that, solve the resulting equation.*

Example 3.4.3 Solve

(1) $2\ln x + 3 = 7$ (2) $\ln(x^2) = \ln(2x+3)$ (3) $-\dfrac{1}{2}\log(x+1) - 3 = 0$ (4) $\ln(x) - \ln(x+3) = \ln 6$

Example 3.4.4 The population of a small town is modeled by the equation $P = 1650e^{0.5t}$ where t is measured in years. In approximately how many years will the town's population reach $20,000$?

Example 3.4.5 The magnitude M of an earthquake is represented by the equation $M = \frac{2}{3}\log\left(\frac{E}{E_0}\right)$ where E is the amount of energy released by the earthquake in joules, and $E_0 = 10^{4.8}$ is the assigned minimal measure released by an earthquake. To the nearest hundredth, if the magnitude of an earthquake is 7.8, how much energy was released?

Example 3.4.6 An account with an initial deposit of $\$6,500$ earns 7.25% annual interest, compounded monthly. After how many years, the balance will be doubled.

Exercises

✍ **Exercise 3.4.1** Solve

(1) $3^{1-x} = 5$ (2) $3^{x-2} = 4^{2x}$ (3) $5 = 10^{3t-2}$ (4) $e^{2x} - 2e^{x} = 15$

✍ **Exercise 3.4.2** Solve

(1) $2\log x - 3 = -1$ (2) $\ln(2x^2) = \ln(5x+3)$ (3) $\frac{1}{2}\log_2(3x-1) = 2$ (4) $\ln(x-1) - \ln(x+1) = 1$

3.5 Exponential and Logarithmic Models

How-to (Exponential Growth or Decay) *The function*

$$A(t) = A_0 e^{kt}$$

is frequently used to model exponential growth (when $k > 0$) or decay (when $k < 0$), where A_0 is the initial quantity.

Example 3.5.1 A population of bacteria doubles every hour. A culture started with 10 bacteria.

(1) After 6 hours how many bacteria will there be?

(2) After how many hours will the population be tripled?

Example 3.5.2 The half-life of carbon-14 is $5,730$ years. A bone fragment is found that contains 20% of its original carbon-14. To the nearest year, how old is the bone?

Example 3.5.3 Sam goes to the doctor and the doctor gives him 15 milligrams of radioactive dye. After 15 minutes, 9 milligrams of dye remain in Sam body. To leave the doctor's office, Sam must pass through a radiation detector that will sound the alarm if more than 2 milligrams of the dye are in his body. How long Sam's visit to the doctor take, assuming he was given the dye as soon as he arrived?

How-to (Newton's Law of Cooling) *The temperature of an object, T, in surrounding air with constant temperature T_s, will behave according to the formula*

$$T(t) = Ae^{kt} + T_s,$$

where t is time, A is the difference between the initial temperature of the object and the surroundings, k is a constant, the continuous rate of cooling of the object.

Example 3.5.4 A cheesecake is taken out of the oven with an ideal internal temperature of $165^\circ\mathrm{F}$, and is placed into a $35^\circ\mathrm{F}$ refrigerator. After 10 minutes, the cheesecake has cooled to $150^\circ\mathrm{F}$. If we must wait until the cheesecake has cooled to $70^\circ\mathrm{F}$ before we eat it, how long will we have to wait?

How-to (Logistic Growth Model) *The logistic growth model is approximately exponential at first, but it has a reduced rate of growth as the output approaches the model's upper bound, called the carrying capacity. The logistic growth of a population over time t is represented by the model*

$$P(t) = \frac{c}{1 + ae^{-bt}},$$

where a, b and c are positive constants, and b is the growth rate, c is the capacity.

Example 3.5.5 The equation $N(t) = \dfrac{500}{1 + 49e^{-0.7t}}$ models the number of people in a small town who have heard a rumor after t days.

(1) What's the population of the small town?

(2) How many people started the rumor?

(3) To the nearest whole number, how many people will have heard the rumor after 3 days?

Exercises

✍ **Exercise 3.5.1** A bacteria culture initially contains 3000 bacteria and doubles every half hour. Find the size of the bacteria population after 80 minutes.

✍ **Exercise 3.5.2** The half-life of tritium-3 is 12.25 years. How long would it take the sample to decay to 20% of its original amount?

✍ **Exercise 3.5.3** A doctor prescribes 125 milligrams of a therapeutic drug that decays by about 30% each hour.

(1) To the nearest hour, what is the half-life of the drug?

(2) How long would it take the drug to decay to 30% of its original amount.

✍ **Exercise 3.5.4** A cup of coffee at $185^{\circ}\mathrm{F}$ is placed into a $60^{\circ}\mathrm{F}$ room. One hour later, the temperature of coffee has dropped to $120^{\circ}\mathrm{F}$. How long will it take for the temperature to drop to $80^{\circ}\mathrm{F}$?

✍ **Exercise 3.5.5** The population of a fish farm in t years is modeled by the equation $P(t) = \dfrac{1000}{1 + 9e^{-0.6t}}$.

(1) What is the initial population of fish?

(2) To the nearest tenth, what is the doubling time for the fish population?

Topic 4. Trigonometric Functions

4.1 Angles

Definition 4.1.1 An **angle** is the union of two rays having a common endpoint. The endpoint is called the **vertex** of the angle, and the two rays are the sides of the angle.

An angle can be created by rotating a ray about its endpoint. The ray at the starting position is called the **initial side** of the angle. The ray at the end position is called the **terminal side** of the angle.

An angle is in **standard position** if its vertex is located at the origin, and its initial side extends along the positive x-axis.

The **measure of an angle** is the amount of rotation from the initial side to the terminal side.

If the angle is measured in a counterclockwise direction from the initial side to the terminal side, the angle is said to be a **positive angle**. If the angle is measured in a clockwise direction, the angle is said to be a **negative angle**.

Definition 4.1.2 An **arc** may be a portion of a full circle, a full circle, or more than a full circle, represented by more than one full rotation.

The length of the arc around an entire circle is called the **circumference** of that circle. An **arc length** is the length of the curve along the arc.

An angle with a vertex at the center of a circle is called a **central angle**.

How-to (Measure of an Angle)

- *One degree is* $\frac{1}{360}$ *of a circular rotation.*
- *One radian is the measure of the central angle of a circle such that the length of the arc between the initial side and the terminal side is equal to the radius of the circle.*
- *A half revolution* 180° *is equivalent to* π *radians.*

Example 4.1.1 Convert each radian measure to degrees and each degree measure to radians.

(1) $\frac{\pi}{3}$ (2) 2 (3) 36° (4) 150°

Definition 4.1.3 **Coterminal angles** are two angles in standard position that have the same terminal side.

The **reference angle** of an angle in the standard position is the acute angle (measured between 0 and $\pi/2$) formed by the terminal side of the angle and the x-axis.

Example 4.1.2 Find a coterminal angle α such that $0^\circ \le \alpha < 360^\circ$ and the reference angle β for the angle $\theta = -45^\circ$.

Example 4.1.3 Find a coterminal angle α such that $0 \le \alpha < 2\pi$ and the reference angle β for the angle $\theta = \dfrac{11\pi}{4}$.

How-to (Arc Length and Sector Area) *Let θ be the radian measure of a central angle in a circle of radius r.*

- *The arc length s of the angle is $s = r\theta$.*
- *The sector area A enclosed by the angle and the arc is $A = \dfrac{1}{2}r^2\theta$.*

Example 4.1.4 Find the arc length along a circle of radius 10 subtended by an angle of 215°.

Example 4.1.5 Find the sector area of a central angle of 150 degree in a circle of radius 12.

Exercises

✍ **Exercise 4.1.1** Find a coterminal angle α in degrees such that $0^\circ \leq \alpha < 360^\circ$ and the reference angle β in radians for the given angle. (1) $\theta = -120^\circ$ (2) $\theta = 400^\circ$ (3) $\theta = \dfrac{8\pi}{3}$ (4) $\theta = -\dfrac{5\pi}{4}$ $\theta = -45^\circ$.

✍ **Exercise 4.1.2** A central angle in a circle of radius is -120°. Find the arc length on the circle and the sector area in the circle that are determined by the angle.

4.2 Unit Circle and Trigonometric Functions

Definition 4.2.1 (Unit Circle and Trigonometric Functions) A unit circle is a circle of radius 1 centered at the origin $(0, 0)$ in the coordinate plane.

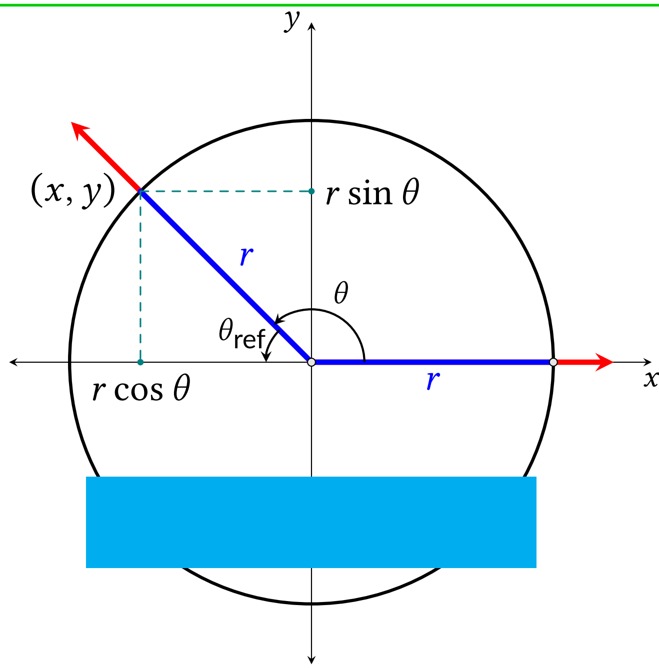
Let θ be a central angle in a unit circle and $P(x, y)$ is the intersection of the terminal side and the unit circle. Then we define $\cos\theta = x$ and $\sin\theta = y$.

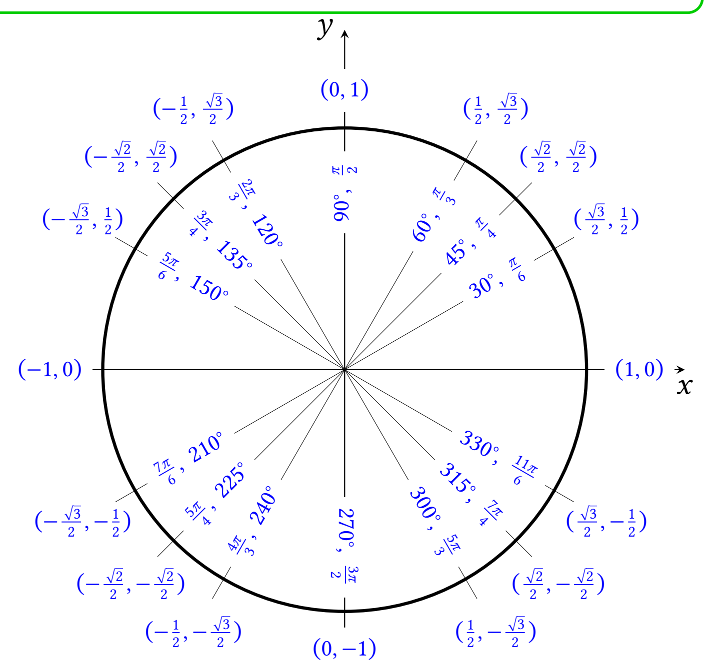
In general, given an angle in the standard position and a point $P(x, y)$ on the terminal side. Assume the distance between P and the origin is r. Then

$$\sin\theta = \frac{y}{r} \qquad \cos\theta = \frac{x}{r}.$$

Other trigonometric functions can be defined using the coordinates and the radius as well as $\sin\theta$ and $\cos\theta$ as follows

$$\tan\theta = \frac{\sin\theta}{\cos\theta} = \frac{y}{x} \qquad \cot\theta = \frac{1}{\tan\theta} = \frac{\cos\theta}{\sin\theta} = \frac{x}{y},$$
$$\sec\theta = \frac{1}{\cos\theta} = \frac{r}{x} \qquad \csc\theta = \frac{1}{\sec\theta} = \frac{1}{\sin\theta} = \frac{r}{y}.$$





Example 4.2.1 Find coordinates of the point that is the intersection of the unit circle and the terminal side of the given angle.

(1) 135° (2) 300° (3) $\dfrac{7\pi}{6}$ (4) $\dfrac{\pi}{3}$

Example 4.2.2 The y-coordinate of a point on the unit circle is $-\frac{\sqrt{2}}{2}$. Find its x-coordinate if the terminal side of the angle is in the third quadrant.

Example 4.2.3 Find the EXACT VALUES of all six trigonometric functions of the central angle θ whose terminal side passes through the point $(-\frac{1}{2}, -\frac{\sqrt{3}}{2})$ on the unit circle.

Example 4.2.4 Find the EXACT VALUES of all six trigonometric functions of the angle θ in the standard position whose terminal side passes through the point $(-3, -4)$.

Example 4.2.5 Use the reference angle to find the EXACT VALUES of all six trigonometric functions of $\frac{5\pi}{6}$

Example 4.2.6 Simplify the expression.

(1) $\dfrac{\sec\theta}{\tan\theta}$.

(2) $\tan t \csc t$

Theorem 4.2.2 (Pythagorean Identity) *For any angle* θ,

$$\sin^2\theta + \cos^2\theta = 1,$$
$$1 + \tan^2\theta = \sec^2\theta,$$
$$1 + \cot^2\theta = \csc^2\theta.$$

Example 4.2.7 Given that $\sec t = -\dfrac{17}{8}$ and $0 < t < \pi$, find the EXACT VALUES of the other five trigonometric functions.

Note (Even or Odd Trigonometric functions)

- *Cosine and secant are even functions:*

$$\cos(-\theta) = \cos\theta \qquad \sec(-\theta) = \sec\theta.$$

- *Sine, tangent, cosecant, and cotangent are odd functions:*

$$\sin(-\theta) = -\sin\theta \qquad \tan(-\theta) = -\tan\theta \qquad \csc(-\theta) = -\csc\theta \qquad \cot(-\theta) = -\cot\theta$$

Example 4.2.8 Find all six trigonometric functions of the angle $-120°$.

Definition 4.2.3 (Periodic Function) A function f is called a **periodic function** if there is number p such that $f(x+p) = f(x)$ for all x. The smalled positive number p such that $f(x+p) = f(x)$ for all x is called the **period** of the function f.

Note *The period of the cosine, sine, secant, and cosecant functions is 2π The period of the tangent and cotangent functions is π.*

Example 4.2.9 Find the EXACT Values of the six trigonometric functions of the angle $\theta = \dfrac{7\pi}{3}$.

Exercises

✍ **Exercise 4.2.1** Find the coordinates of the point on the unit circle and the terminal side of the given angle.

(1) $\theta = 30^\circ$ (2) $\theta = 225^\circ$ (3) $\theta = \frac{3\pi}{4}$ (4) $\theta = \frac{11\pi}{6}$

✍ **Exercise 4.2.2** Find all six trigonometric functions of the angle in the standard position whose terminal side passing through the given point.

(1) $(-1, 2)$ (2) $(\frac{\sqrt{3}}{2}, -\frac{1}{2})$ (3) $(-4, -3)$

✍ **Exercise 4.2.3** Find all six trigonometric functions of each angle.

(1) $A = -45^\circ$ (2) $B = \frac{4\pi}{3}$ (3) $C = -\frac{5\pi}{6}$

✍ **Exercise 4.2.4** Simplify the expression.

(1) $\dfrac{\cot\theta}{\csc\theta}$ (2) $\sec\theta\tan\theta\cos^2\theta$

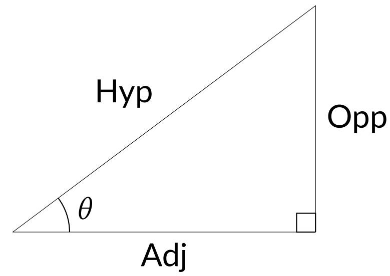
✍ **Exercise 4.2.5** Given that $\tan\theta = -2$ and $-\dfrac{\pi}{2} < \theta < \dfrac{\pi}{0}$, find the EXACT VALUES of the other five trigonometric functions.

4.3 Right Triangle Trigonometry

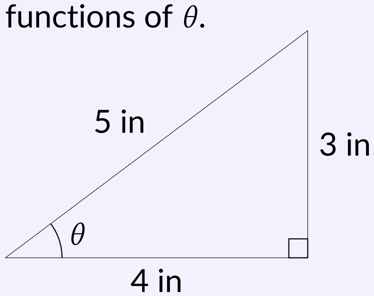
Definition 4.3.1 Given a right triangle with an acute angle θ, the six **trigonometric functions** are defined as follows.

$$\sin\theta = \frac{\text{Opp}}{\text{Hyp}} \qquad \cos\theta = \frac{\text{Adj}}{\text{Hyp}} \qquad \tan\theta = \frac{\text{Opp}}{\text{Adj}}$$

$$\csc\theta = \frac{\text{Hyp}}{\text{Opp}} \qquad \sec\theta = \frac{\text{Hyp}}{\text{Adj}} \qquad \cot\theta = \frac{\text{Adj}}{\text{Opp}}$$

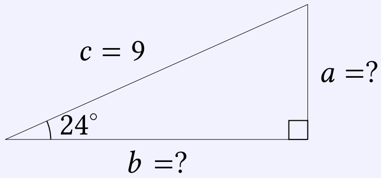


Example 4.3.1 Consider the right triangle bellow. The adjacent side of one of the acute angles θ is 4 in, and the opposite side is 3 in, and the hypotenuse is 5 in. Find all values of trigonometric functions of θ.



Example 4.3.2 In triangle $\triangle ABC$, if $\angle C = 90^\circ$, $AB = 19$ cm and $\angle B = 23^\circ$, determine the length of AC and the length of BC to the nearest tenth of a centimeter.

Example 4.3.3 Find sides a and b in the following right triangle. The standard convention is that the lower case letter is the side opposite the angle with the corresponding capital letter.



Example 4.3.4 The angle of elevation to the top of a tall tree is 55° when measured at a point 30 feet from the base. Assume the ground is flat. How tall is the tree?

Example 4.3.5 A lighthouse is 200 feet above the sea level. A boat was spotted from the top of the lighthouse at an angle of depression of 5°. How far was the boat from the lighthouse?

Example 4.3.6 To estimate the height of a building, two measurements are taken. The first measurement shows an angle of elevation to the top of the building as 51°. The second measurement, taken 50 feet closer to the base of the building, yields an angle of elevation of 77°. From the measurements, estimate the height of the building. **Round to the nearest foot.**

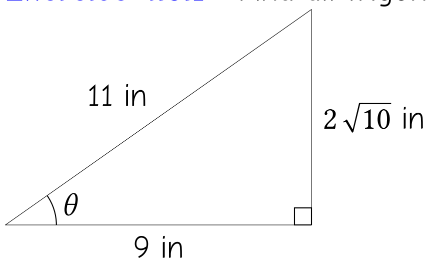
Theorem 4.3.2 (Cofunction Identities) *Given an angle θ measured in radians, we have the following cofunction identities.*

$$\begin{aligned} \cos\theta &= \sin\left(\frac{\pi}{2} - \theta\right) & \sin\theta &= \cos\left(\frac{\pi}{2} - \theta\right) \\ \cot\theta &= \tan\left(\frac{\pi}{2} - \theta\right) & \tan\theta &= \cot\left(\frac{\pi}{2} - \theta\right) \\ \csc\theta &= \sec\left(\frac{\pi}{2} - \theta\right) & \sec\theta &= \csc\left(\frac{\pi}{2} - \theta\right) \end{aligned}$$

Example 4.3.7 If $\sin t = \dfrac{5}{12}$, find $\cos(\frac{\pi}{2} - t)$.

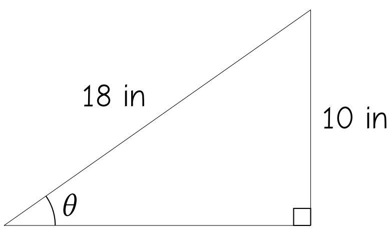
Exercises

Exercise 4.3.1 Find all trigonometric functions of the angle θ in the right triangle given below.

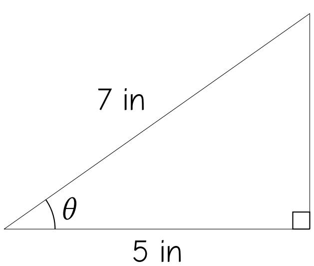


Exercise 4.3.2 Find $\sin\theta$, $\cos\theta$ and $\tan\theta$ of the angle θ given in the figure.

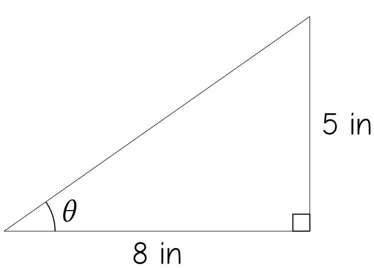
(1)



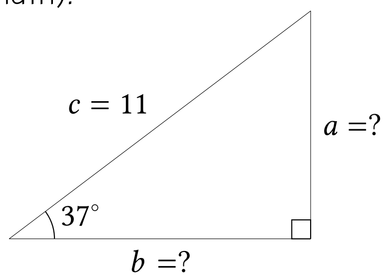
(2)



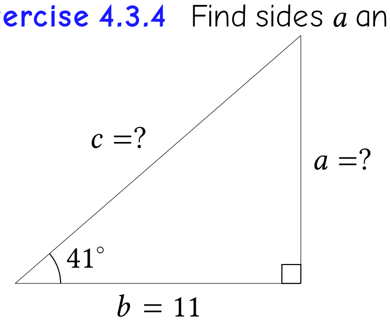
(3)



✍ **Exercise 4.3.3** Find sides a and b in the following right triangle (round to the nearest thousandth).



✍ **Exercise 4.3.4** Find sides a and c in the following right triangle (round to the nearest thousandth).



✍ **Exercise 4.3.5** In triangle $\triangle ABC$, if $\angle C = 90^\circ$, $AC = 52$ cm and $\angle B = 37^\circ$, determine the length of AB and the length of BC to the nearest tenth of a centimeter.

✍ **Exercise 4.3.6** A hot air balloon hovers above the ground at a height of 1000 feet. A person on the ground sees the balloon at an angle of elevation of $27°$. What is the distance between the balloon and the person? (Round to the nearest foot.)

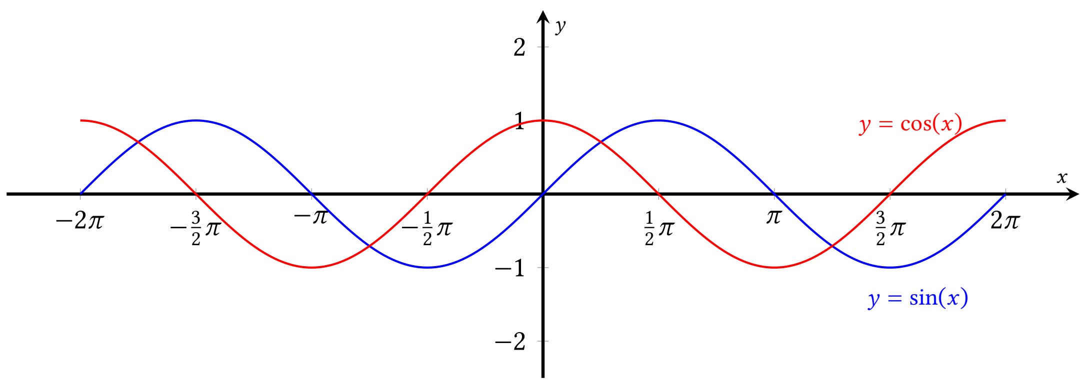
✍ **Exercise 4.3.7** A jet takes off at a $20°$ angle. The runway from takeoff is 800 meters long. What is the altitude of the airplane when it flies over the end of the runway? (Round to the nearest tenth of a meter)

✍ **Exercise 4.3.8** If $\cos\alpha = \dfrac{12}{13}$, find $\sin\left(\dfrac{\pi}{2} - \alpha\right)$.

4.4 Graphs of Sine and Cosine

Note (Charateristics of sine and cosine functions)

- *They are periodic functions with a period of* 2π.
- *The domain of each function is* $(-\infty, \infty)$.
- *The range of each function is* $[-1, 1]$.
- *The sine function* $y = \sin x$ *is odd and the graph is symmetric about the origin.*
- *The cosine function* $y = \cos x$ *is even and the graph is symmetric about the* y*-axis.*
- *The sine function has the* y*-intercept* $(0, 0)$ *and* x*-intercepts* $(k\pi, 0)$, *where* k *is any integer.*
- *The cosine function* $y = \sin x$ *has the* y*-intercept* $(0, 1)$ *and* x*-intercepts* $\left(k\pi + \frac{\pi}{2}, 0\right)$, *where* k *is any integer.*
- *The sine function has the global (also local) maximum* $1 = \sin\left(\frac{(2k+1)\pi}{2}\right)$ *and the global also local minimum* $-1 = \sin\left(\frac{(2k-1)\pi}{2}\right)$, *where* k *is any integer..*
- *The cosine function* $y = \cos x$ *has the global (also local) maximum* $1 = \cos(2k\pi)$ *and the global also local minimum* $-1 = \cos((2k+1)\pi)$, *where* k *is any integer.*
- *By the cofunction identity* $\cos x = \sin(x + \frac{\pi}{2})$, *the graph of* $y = \cos x$ *can be obtained by shifting the graph of* $y = \cos x$ *horizontally* $-\frac{\pi}{2}$ *units.*



Sine and Coside in two cycles: $[-2\pi, 2\pi]$

Definition 4.4.1 A **sinusoidal function** is a function f that is defined by $f(x) = A\sin(Bx - C) + D$ or $f(x) = A\cos(Bx - C) + D$. The horizontal line $y = D$ is called the **midline**. The **amplitude** of f is maximal distance that a value of f can be above or below the midline, that is

$$\text{amplitude} = \frac{1}{2}|f_{\max} - f_{\min}|.$$

How-to (Characterize Sinusoidal Function) *Given a sinusoidal function* $y = A\sin(Bx - C) + D$ *or* $y = A\cos(Bx - C) + D$, *or equivalently,* $y = A\sin(B(x - \frac{C}{B})) + D$ *or* $y = A\cos(B(x - \frac{C}{B})) + D$,

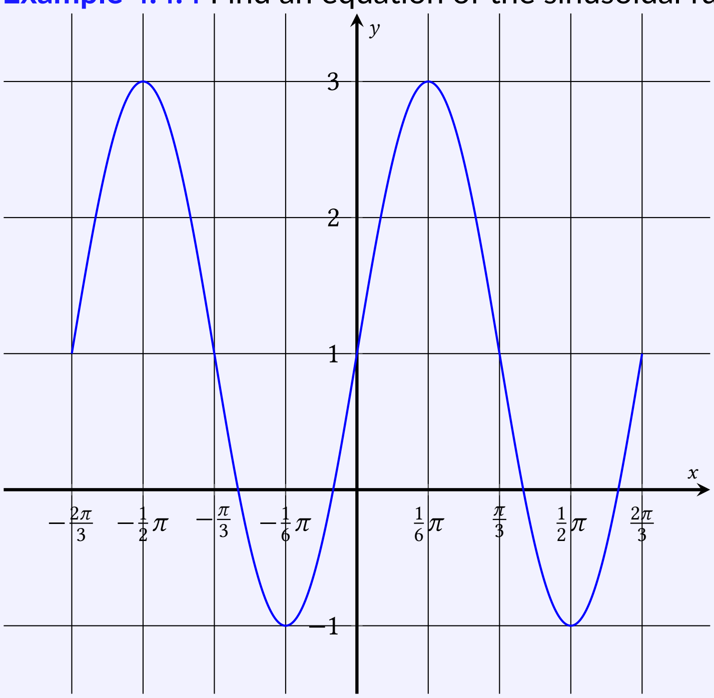
- *the amplitude is* $|A|$;
- *the period is* $\dfrac{2\pi}{B}$;
- *the phase shift is* $\dfrac{C}{B}$;
- *the midline is* $y = D$.

Example 4.4.1 Determine the midline, amplitude, period, and phase shift of the function $y = 3\sin(2x) + 1$.

Example 4.4.2 Sketch a graph of $f(x) = -2\sin\left(\frac{\pi x}{2}\right)$.

Example 4.4.3 Given $y = -2\cos\left(\frac{\pi}{2}x + \pi\right) + 3$, determine the amplitude, period, phase shift, and midline. Then graph the function.

Example 4.4.4 Find an equation of the sinusoidal function defined by the following graph.

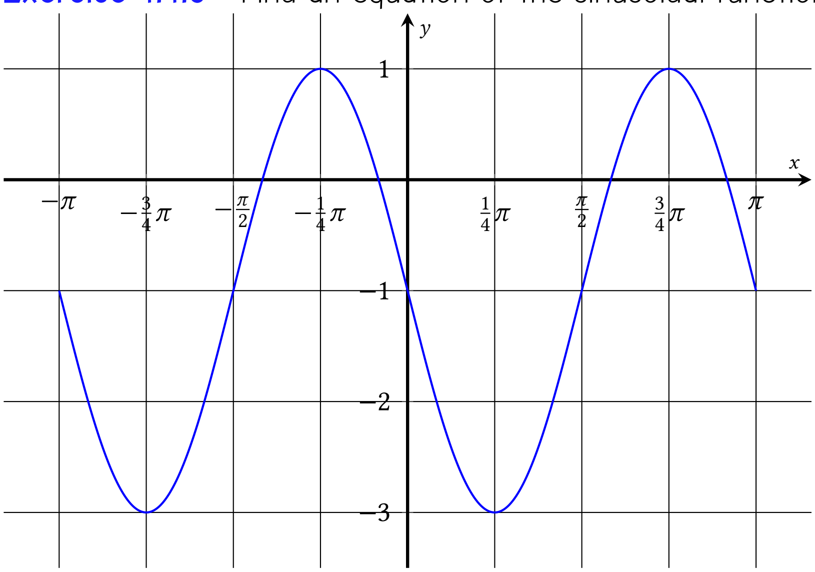


Exercises

✍ **Exercise 4.4.1** Determine the midline, amplitude, period, and phase shift of the function $y = 2\cos(2\pi x - \pi) - 1$.

✍ **Exercise 4.4.2** Given $y = -3\sin\left(\frac{\pi}{2}x - \pi\right) + 2$, determine the amplitude, period, phase shift, and midline. Then graph the function.

✍ **Exercise 4.4.3** Find an equation of the sinusoidal function defined by the following graph.



4.5 Graph of Other Trigonometric Functions

How-to (Graph of $y = A\tan(Bx)$)

- *The stretching factor is* $|A|$.
- *The period is* $P = \dfrac{\pi}{|B|}$.
- *The domain consists of all real numbers* x *such that* $x \neq \dfrac{(2k+1)\pi}{2|B|}$ *for all integer* k.
- *The range is* $(-\infty, \infty)$.
- *The vertical asymptote* $x = \dfrac{(2k+1)\pi}{2|B|}$.
- *The function is an odd function.*
- *The* y*-intercept is* $(0, 0)$.
- *The* x*-intercepts are* $\left(\dfrac{k\pi}{|B|}, 0\right)$.

Example 4.5.1 Sketch a graph of one period of the function $y = \dfrac{1}{2}\tan\left(\dfrac{\pi}{2}x\right)$.

Note *The graph of the cotangent function can be obtained from the graph of a tangent function by horizontal shift of* $-\dfrac{\pi}{2B}$ *units.*

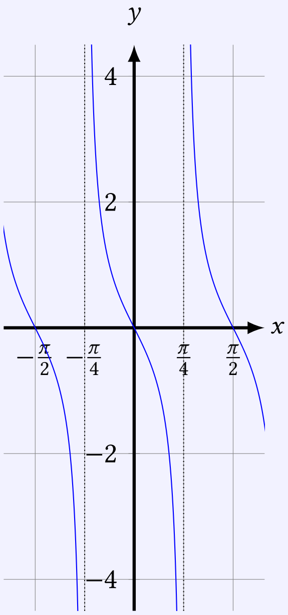
How-to (Graph of $y = A\sec(Bx)$)

- *The stretching factor is* $|A|$.
- *The period is* $\dfrac{2\pi}{|B|}$.
- *The domain consists of all real numbers* x *such that* $x \neq \dfrac{(2k+1)\pi}{2|B|}$, *where* k *is an integer.*
- *The range is* $(-\infty, -|A|] \cup [|A|, \infty)$
- *The vertical asymptotes are* $x = \dfrac{(2k+1)\pi}{2|B|}$, *where* k *is an integer.*
- *The function is an even function.*

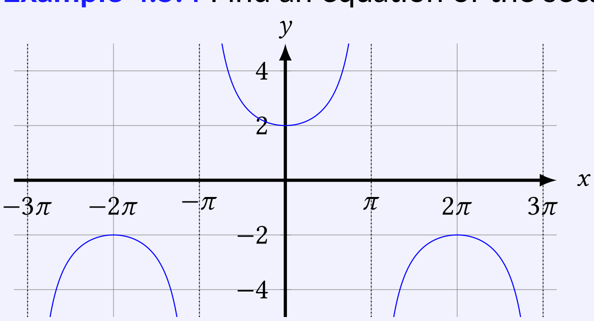
Example 4.5.2 Sketch a graph of $f(x) = 2\sec(\pi x)$ in one period.

Note *The graph of the cosecant function can be obtained from the graph of a secant function by a horizontal shift of* $\dfrac{\pi}{2B}$ *units.*

Example 4.5.3 Find an equation of the tangent function defined by the following graph.



Example 4.5.4 Find an equation of the secant function defined by the following graph.

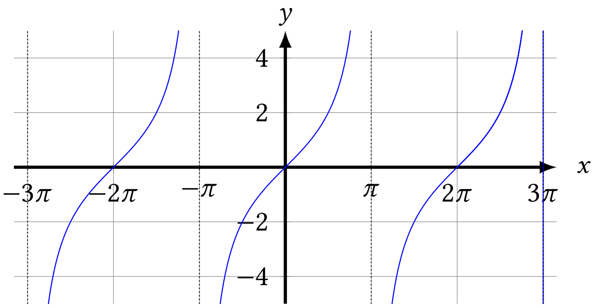


Exercises

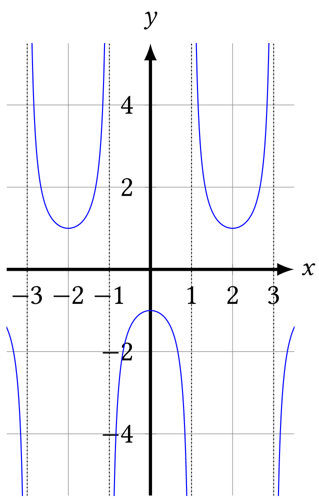
✍ **Exercise 4.5.1** Sketch a graph of $f(x) = 3\tan\left(\frac{\pi}{6}x\right)$ in one period.

✍ **Exercise 4.5.2** Sketch a graph of $f(x) = -2\sec(4x)$ in one period.

✍ **Exercise 4.5.3** Find an equation of the tangent function defined by the following graph.



✍ **Exercise 4.5.4** Find an equation of the secant function defined by the following graph.



4.6 Inverse Trigonometric Functions

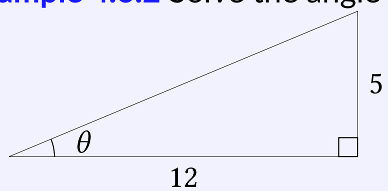
Definition 4.6.1 On restricted domains, we can define the inverse trigonometric functions.

- The inverse sine function $y = \sin^{-1} x$ means $x = \sin y$. The inverse sine function is sometimes called the **arcsine** function , and notated $\arcsin x$. $y = \sin^{-1} x$ has domain $[-1, 1]$ and range $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$.
- The inverse cosine function $y = \cos^{-1} x$ means $x = \cos y$. The inverse cosine function is sometimes called the **arccosine** function, and notated $\arccos x$. $y = \cos^{-1} x$ has domain $[-1, 1]$ and range $[0, \pi]$.
- The inverse tangent function $y = \tan^{-1} x$ means $x = \tan y$. The inverse tangent function is sometimes called the **arctangent** function, and notated $\arctan x$. $y = \tan^{-1} x$ has domain $(-\infty, \infty)$ and range $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$.

Example 4.6.1 Evaluate each of the following.

(1) $\sin^{-1}\left(-\frac{\sqrt{2}}{2}\right)$ (2) $\cos^{-1}\left(-\frac{\sqrt{3}}{2}\right)$ (3) $\tan^{-1}(1)$

Example 4.6.2 Solve the angle θ (rounded to the hundredth radian) from the given right triangle.



How-to (The Composition of a Trigonometric Function and an Inverse Trigonometric Function)

From the definition of inverse function, we know

$$
\begin{aligned}
\sin(\sin^{-1} x) &= x && \textit{for } -1 \le x \le 1\\
\cos(\cos^{-1} x) &= x && \textit{for } -1 \le x \le 1\\
\tan(\tan^{-1} x) &= x && \textit{for } -\infty < x < \infty\\
\sin^{-1}(\sin x) &= x && \textit{only for } -\frac{\pi}{2} \le x \le \frac{\pi}{2}\\
\cos^{-1}(\cos x) &= x && \textit{only for } 0 \le x \le \pi\\
\tan^{-1}(\tan x) &= x && \textit{only for } -\frac{\pi}{2} < x < \frac{\pi}{2}.
\end{aligned}
$$

The trigonometric identities may also need to when the functions in the composition are not inverse to each other.

Example 4.6.3 Evaluate the following.

(1) $\sin^{-1}\left(\sin\left(\frac{\pi}{3}\right)\right)$

(2) $\cos^{-1}\left(\cos\left(-\frac{\pi}{3}\right)\right)$

Example 4.6.4 Evaluate $\sin^{-1}\left(\cos\left(\frac{13\pi}{6}\right)\right)$.

Example 4.6.5 Find an exact value for $\sin\left(\cos^{-1}\left(\frac{4}{5}\right)\right)$

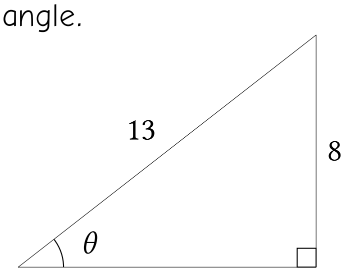
Example 4.6.6 Find an exact value for $\sin\left(\tan^{-1}\left(\frac{7}{4}\right)\right)$.

Exercises

✍ **Exercise 4.6.1** Evaluate each of the following.

(1) $\sin^{-1}\left(\frac{\sqrt{3}}{2}\right)$ (2) $\cos^{-1}\left(-\frac{1}{2}\right)$ (3) $\tan^{-1}(-\sqrt{3})$

✍ **Exercise 4.6.2** Solve the angle θ (rounded to the hundredth radian) from the given right triangle.



✍ **Exercise 4.6.3** Evaluate the following.

(1) $\sin^{-1}\left(\sin\left(\frac{\pi}{6}\right)\right)$ (2) $\cos^{-1}\left(\cos\left(-\frac{\pi}{4}\right)\right)$

✍ **Exercise 4.6.4** Evaluate $\cos^{-1}\left(\sin\left(\frac{11\pi}{3}\right)\right)$.

✍ **Exercise 4.6.5** Find an exact value for $\sin\left(\cos^{-1}\left(\frac{3}{5}\right)\right)$

✍ **Exercise 4.6.6** Find an exact value for $\cos\left(\tan^{-1}\left(\frac{5}{4}\right)\right)$.

5.1 Simplifying Trigonometric Expressions

Note (Fundamental Identities)

Pythagorean identities

$$\sin^2 x + \cos^2 x = 1$$
$$1 + \tan^2 x = \sec^2 x$$
$$1 + \cot^2 x = \csc^2 x$$

Even-odd identities

$$\sin(-x) = -\sin x$$
$$\cos(-x) = \cos x$$
$$\tan(-x) = -\tan x$$

Reciprocal identities

$$\cot x = \frac{1}{\tan x}$$
$$\csc x = \frac{1}{\sin x}$$
$$\sec x = \frac{1}{\cos x}$$

Quotient identities

$$\tan x = \frac{\sin x}{\cos x}$$
$$\cot x = \frac{\cos x}{\sin x}$$

Example 5.1.1 Verify the trigonometric identity.

(1) $\tan\theta\cos\theta = \sin\theta$ (2) $\dfrac{\sec^2\theta - 1}{\sec^2\theta} = \sin^2\theta$

Example 5.1.2 Simplify the trigonometric identity.

(1) $\dfrac{\sin^2(-\theta) - \cos^2(-\theta)}{\sin(-\theta) - \cos(-\theta)}$ (2) $(1 - \cos^2 x)(1 + \cot^2 x)$

Exercises

✍ **Exercise 5.1.1** Verify the trigonometric identity.

(1) $\dfrac{\tan x}{\sec x}\sin(-x) = \cos^2 x$ (2) $\dfrac{\cos^2\theta - \sin^2\theta}{1 - \tan^2\theta} = \sin^2\theta$ (3) $\dfrac{\sec(-x)}{\tan x + \cot x} = -\sin(-x)$

✍ **Exercise 5.1.2** Simplify the trigonometric identity.

(1) $\tan x \sin x + \sec x \cos^2 x$ (2) $\dfrac{\cot t + \tan t}{\sec(-t)}$ (3) $\dfrac{1 - \cos^2 x}{\tan^2 x} + 2\sin^2 x$

5.2 Formulas of Angle Addition

Theorem 5.2.1 (Sine and Cosine of Sum or Difference of Angles[5])

$$\sin(\alpha + \beta) = \sin\alpha\cos\beta + \sin\alpha\cos\beta$$
$$\cos(\alpha + \beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta$$
$$\sin(\alpha - \beta) = \sin\alpha\cos\beta - \sin\alpha\cos\beta$$
$$\cos(\alpha - \beta) = \cos\alpha\cos\beta + \sin\alpha\sin\beta$$

Example 5.2.1 Find the exact value.

(1) $\cos(75^\circ)$

(2) $\sin(135^\circ)$

Example 5.2.2 Find the exact value of $\sin\left(\cos^{-1}\left(\frac{1}{2}\right) + \sin^{-1}\left(\frac{3}{5}\right)\right)$.

[5]One proof is to use Euler formula: $e^{i\alpha} = \cos\alpha + i\sin\alpha$.

Example 5.2.3 Given $\sin\alpha = \frac{3}{5}, \quad 0 < \alpha < \frac{\pi}{2}$, and $\cos\beta = -\frac{5}{13}, \quad \pi < \beta < \frac{3\pi}{2}$, find
(1) $\sin(\alpha+\beta)$ (2) $\cos(\alpha-\beta)$ (3) $\tan(\alpha+\beta)$ (4) $\csc(\alpha-\beta)$

Example 5.2.4 Prove the following identities using the identities for sum/difference of angles.
(1) $\sin(\frac{\pi}{2} - x) = \cos x$ (2) $\sin(\pi - x) = \sin x$ (3) $\cos(\pi - x) = -\cos x$

Example 5.2.5 Verify the identity $\sin(\alpha + \beta) + \sin(\alpha - \beta) = 2 \sin\alpha \cos\beta$.

Example 5.2.6 Verify the identity $\cos(2\alpha) = \cos^2\alpha - \sin^2\alpha$.

Exercises

✍ **Exercise 5.2.1** Find the exact value.

(1) $\cos(75°)$ (2) $\sin(135°)$

✍ **Exercise 5.2.2** Find the exact value of $\cos\left(\cos^{-1}\left(\frac{1}{3}\right) - \sin^{-1}\left(\frac{4}{5}\right)\right)$.

✍ **Exercise 5.2.3** Given $\sin\alpha = -\frac{4}{5}$, $\pi < \alpha < \frac{3\pi}{2}$, and $\cos\beta = \frac{12}{13}$, $0 < \beta < \frac{\pi}{2}$, find

(1) $\sin(\alpha - \beta)$ (2) $\cos(\alpha + \beta)$ (3) $\cot(\alpha - \beta)$

✍ **Exercise 5.2.4** Prove the following identities using the identities for sum/difference of angles.
(1) $\cos(x+\frac{\pi}{2})=\sin x$ (2) $\sin(x-\pi)=-\sin x$ (3) $\cos(x+\pi)=-\cos x$

✍ **Exercise 5.2.5** Verify the identity $\cos(\alpha-\beta)-\cos(\alpha+\beta)=2\sin\alpha\sin\beta$.

✍ **Exercise 5.2.6** Verify the identity $\sin(2\alpha)=2\sin\alpha\cos\alpha$.

5.3 Double and Half Angle Formulas

Corollary 5.3.1 (Double Angle Identities)

$$\begin{aligned}\sin(2\alpha) &= 2\sin\alpha\cos\alpha \\ \cos(2\alpha) &= \cos^2\alpha - \sin^2\alpha \\ &= 2\cos^2\alpha - 1 \\ &= 1 - 2\sin^2\alpha\end{aligned}$$

Corollary 5.3.2 (Half Angle Identities)

$$\begin{aligned}\sin^2\left(\frac{\theta}{2}\right) &= \frac{1-\cos\theta}{2} \\ \cos^2\left(\frac{\theta}{2}\right) &= \frac{1+\cos\theta}{2}\end{aligned}$$

Example 5.3.1 Find the exact value.

(1) $\sin\left(2\cos^{-1}\left(\frac{3}{5}\right)\right)$

(2) $\tan\left(2\sin^{-1}\left(\frac{3}{5}\right)\right)$

Example 5.3.2 Verify the identity $\cos^4\theta - \sin^4\theta = \cos(2\theta)$.

Example 5.3.3 Verify the identity: $\tan(2\theta) = 2\cot\theta - \tan\theta$

Example 5.3.4 Write an equivalent expression for $\cos^4 x$ that does not involve any powers of sine or cosine greater than 1.

Example 5.3.5 Find $\sin 15^\circ$ and $\cos 15^\circ$.

Example 5.3.6 Given that $\tan\alpha = \dfrac{8}{15}$ and α lies in quadrant III, find the exact value of the following:

(1) $\sin\left(\dfrac{\alpha}{2}\right)$ (2) $\cos\left(\dfrac{\alpha}{2}\right)$ (3) $\tan\left(\dfrac{\alpha}{2}\right)$

Exercises

✍ **Exercise 5.3.1** Find the exact value.

(1) $\cos\left(2\sin^{-1}\left(\frac{4}{5}\right)\right)$ (2) $\tan\left(2\cos^{-1}\left(\frac{4}{5}\right)\right)$

✍ **Exercise 5.3.2** Verify the identity $(\cos\theta - \sin\theta)^2 = 1 - \sin(2\theta)$.

✍ **Exercise 5.3.3** Write an equivalent expression for $\sin^4 x$ that does not involve any powers of sine or cosine greater than 1.

✍ **Exercise 5.3.4** Given that $\sin\alpha = -\dfrac{4}{5}$ and α lies in quadrant IV, find the exact value of $\tan\left(\dfrac{\alpha}{2}\right)$.

5.4 Sum-to-Product and Product-to-Sum Formulas

Corollary 5.4.1 (The Product-to-Sum Formulas)

$$\cos\alpha\cos\beta = \frac{1}{2}[\cos(\alpha-\beta)+\cos(\alpha+\beta)]$$
$$\sin\alpha\cos\beta = \frac{1}{2}[\sin(\alpha+\beta)+\sin(\alpha-\beta)]$$
$$\sin\alpha\sin\beta = \frac{1}{2}[\cos(\alpha-\beta)-\cos(\alpha+\beta)]$$

Corollary 5.4.2 (The Sum-to-Produc Formulas)

$$\cos\alpha+\cos\beta = 2\cos\left(\frac{\alpha+\beta}{2}\right)\cos\left(\frac{\alpha-\beta}{2}\right)$$
$$\sin\alpha+\sin\beta = 2\sin\left(\frac{\alpha+\beta}{2}\right)\cos\left(\frac{\alpha-\beta}{2}\right)$$
$$\cos\alpha-\cos\beta = 2\sin\left(\frac{\alpha+\beta}{2}\right)\sin\left(\frac{\alpha-\beta}{2}\right)$$

Example 5.4.1 Write the following product as a sum

(1) $2\cos\left(\frac{7x}{2}\right)\cos\left(\frac{3x}{2}\right)$ (2) $\sin(3\theta)\cos(5\theta)$

Example 5.4.2 Write the following difference or sum expression as a product. (1) $\sin(3\theta)-\sin\theta$
(2) $\cos(2\theta)+\cos(4\theta)$ (3) $\cos(4\theta)-\cos(2\theta)$ (4) $\sin\theta-\cos\theta$

Example 5.4.3 Evaluate (1) $\cos(15^\circ) - \cos(75^\circ)$ (2) $\sin(15^\circ) + \sin(45^\circ)$ (3) $\sin(45^\circ) - \cos(135^\circ)$

Example 5.4.4 Prove the identity:

$$\frac{\cos(4t) - \cos(2t)}{\sin(4t) + \sin(2t)} = -\tan t$$

Exercises

✍ **Exercise 5.4.1** Write the following product as a sum

(1) $\sin\left(\frac{\theta}{2}\right)\cos\left(\frac{5\theta}{2}\right)$ (2) $\sin(4\theta)\sin(2\theta)$

✍ **Exercise 5.4.2** Write the following difference or sum expression as a product. (1) $\sin(5\theta)-\sin\theta$

(2) $\cos(\theta)+\sin(\theta)$ (3) $\cos(3\theta)+\cos(5\theta)$

✍ **Exercise 5.4.3** Evaluate (1) $\sin(75°) - \cos(75°)$ (2) $\sin(45°) + \sin(135°)$

✍ **Exercise 5.4.4** Prove the identity $\sin x + \sin(3x) = 4 \sin x \cos^2 x$.

5.5 Solving Trigonometric Equations

Example 5.5.1 Find all possible exact solutions for the equation.

(1) $\cos\theta = \dfrac{1}{2}$ (2) $\sin\theta = \dfrac{1}{2}$ (3) $\tan\theta = \dfrac{\sqrt{3}}{3}$

Example 5.5.2 Solve the equation exactly: $2\cos\theta - 3 = -5$, $0 \le \theta < 2\pi$.

Example 5.5.3 Solve the equation exactly: $2\sin^2\theta - 1 = 0$, $0 \le \theta < 2\pi$.

Example 5.5.4 Solve the equation exactly: $\cos^2\theta + 3\cos\theta - 1 = 0$, $0 \le \theta < 2\pi$.

Example 5.5.5 Solve the equation exactly over the interval $0 \le x < 2\pi$

$$\cos x\cos(2x) + \sin x\sin(2x) = \frac{\sqrt{3}}{2}.$$

Example 5.5.6 Solve the equation exactly: $\cos(2\theta) = \cos\theta$.

Example 5.5.7 Solve the equation exactly: $2\cos^2\theta - 3\sin\theta = 3$.

Example 5.5.8 Solve the equation quadratic in form exactly: $2\sin^2\theta - 3\sin\theta + 1 = 0$, $0 \le \theta < 2\pi$

Exercises

✍ **Exercise 5.5.1** Solve the equation exactly: $2\sin\theta\cos\theta - \sqrt{3} = 0,\ 0 \leq \theta < 2\pi$.

✍ **Exercise 5.5.2** Solve the equation exactly: $\cos^2\theta + 3\cos\theta - 1 = 0,\ 0 \leq \theta < 2\pi$.

✍ **Exercise 5.5.3** Solve the equation quadratic in form exactly: $2\sin^2\theta - 3\sin\theta + 1 = 0,\ 0 \leq \theta < 2\pi$

✍ **Exercise 5.5.4** Solve the equation exactly over the interval $0 \leq x < 2\pi$

$$\sin x \cos(2x) + \cos x \sin(2x) = \frac{1}{2}.$$

✍ **Exercise 5.5.5** Solve the equation exactly: $sin(2\theta) = cos\theta$.

✍ **Exercise 5.5.6** Solve the equation exactly: $2\sin^2\theta = 3\cos\theta - 3$.

6.1 Law of Sine

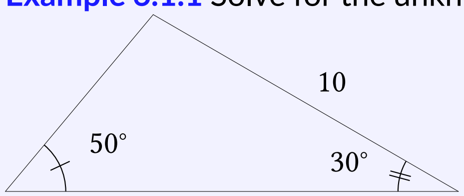
Theorem 6.1.1 (Law of Sine) *Given a triangle $\triangle ABC$ with sides of lengths a, b, and c opposite to angles A, B, and C, respectively, then*

$$\frac{\sin A}{a} = \frac{\sin B}{b} = \frac{\sin C}{c} \quad \text{or} \quad \frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$$

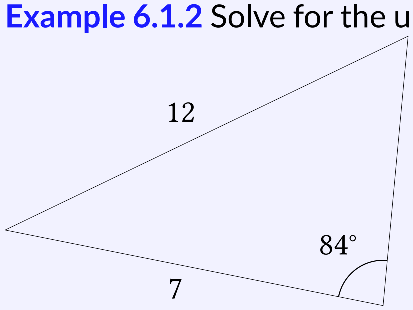
Corollary 6.1.2 (Area of Triangle) *Given a triangle $\triangle ABC$ with sides of lengths a, b, and c opposite to angles A, B, and C, respectively, then the area S of the triangle is*

$$S = \frac{1}{2}ab\sin C = \frac{1}{2}ac\sin b = \frac{1}{2}bc\sin A.$$

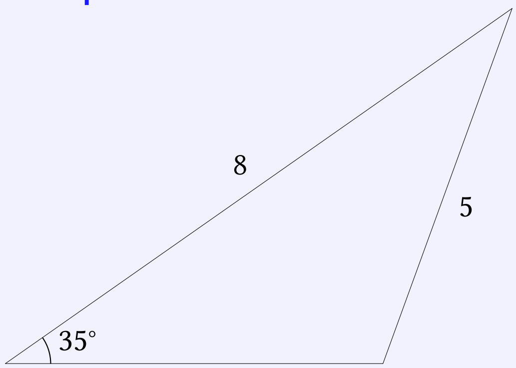
Example 6.1.1 Solve for the unknown side and angles. Round your answers to the nearest tenth.



Example 6.1.2 Solve for the unknown side and angles. Round your answers to the nearest tenth.



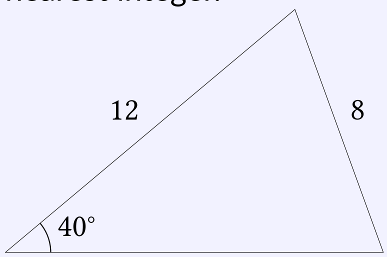
Example 6.1.3 Solve for the unknown side and angles. Round your answers to the nearest tenth.



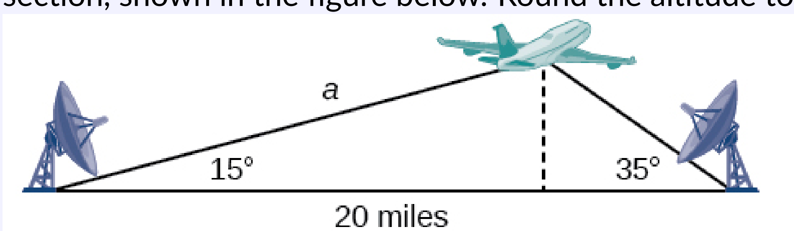
Example 6.1.4 Find all possible triangles if one side has length 4 opposite an angle of 50°, and a second side has length 10.

Example 6.1.5 Find the area of a triangle with sides $a = 90$, $b = 52$, and the angle $C = 102^{\circ}$ formed by those two sides. Round the area to the nearest integer.

Example 6.1.6 Find the area of a triangle shown in the figure below. Round the area to the nearest integer.

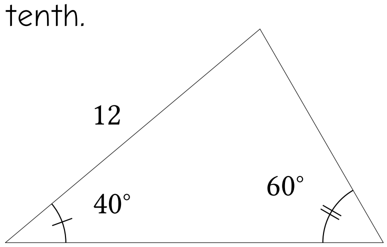


Example 6.1.7 Find the altitude of the aircraft in the problem introduced at the beginning of this section, shown in the figure below. Round the altitude to the nearest tenth of a mile.

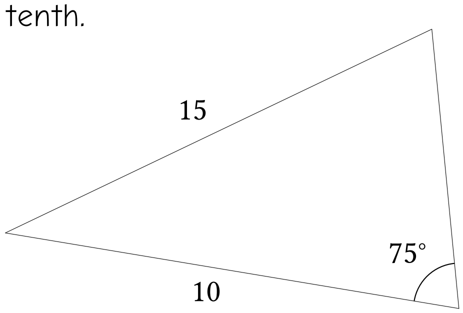


Exercises

✍ **Exercise 6.1.1** Solve for the unknown side and angles. Round your answers to the nearest tenth.

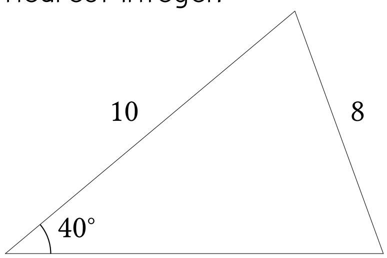


✍ **Exercise 6.1.2** Solve for the unknown side and angles. Round your answers to the nearest tenth.

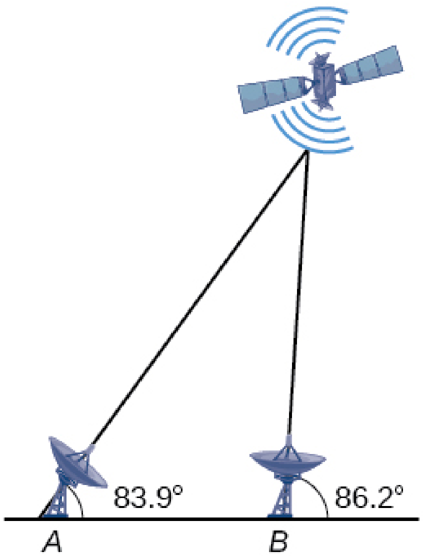


✍ **Exercise 6.1.3** Find all possible triangles if one side has length 5 opposite an angle of 70°, and a second side has length 9.

✍ **Exercise 6.1.4** Find the area of a triangle shown in the figure below. Round the area to the nearest integer.



✍ **Exercise 6.1.5** The Figure below shows a satellite orbiting Earth. The satellite passes directly over two tracking stations A and B, which are 69 miles apart. When the satellite is on one side of the two stations, the angles of elevation at A and B are measured to be $86.2°$ and $83.9°$ respectively. How far is the satellite from station A and how high is the satellite above the ground? Round answers to the nearest whole mile.



6.2 Law of Cosine

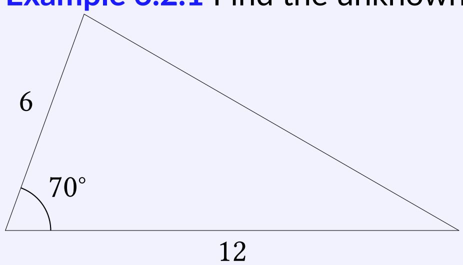
Theorem 6.2.1 (Law of Cosine) *Given a triangle $\triangle ABC$ with sides of lengths a, b, and c opposite to angles A, B, and C, respectively, then*

$$\begin{aligned} a^2 &= b^2 + c^2 - 2bc\cos A \\ b^2 &= a^2 + c^2 - 2ac\cos B \\ c^2 &= a^2 + b^2 - 2ab\cos C \end{aligned}$$

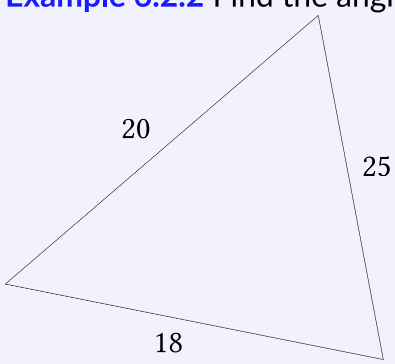
or

$$\begin{aligned} \cos A &= \frac{b^2 + c^2 - a^2}{2bc} \\ \cos B &= \frac{a^2 + c^2 - b^2}{2ac} \\ \cos C &= \frac{a^2 + b^2 - c^2}{2ab}. \end{aligned}$$

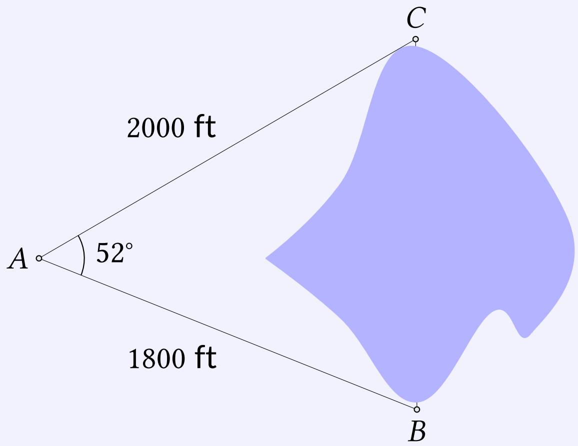
Example 6.2.1 Find the unknown side and angles of the triangle.



Example 6.2.2 Find the angles in the triangle. Round your answers to the nearest tenth.



Example 6.2.3 To find the distance across a small lake, a surveyor has taken the measurements shown in the figure below. Find the distance across the lake using this information. Round your answers to the nearest tenth.

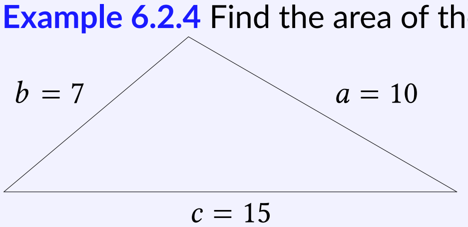


Theorem 6.2.2 (Heron's Formula) *The area of oblique triangles in which sides* a, b, *and* c *are known.*

$$\text{Area} = \sqrt{s(s-a)(s-b)(s-c)},$$

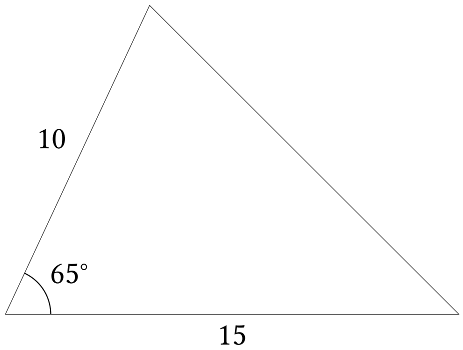
where $s = \dfrac{(a+b+c)}{2}$ *is one half of the perimeter of the triangle, sometimes called the semi-perimeter.*

Example 6.2.4 Find the area of the triangle in the figure below using Heron's formula.

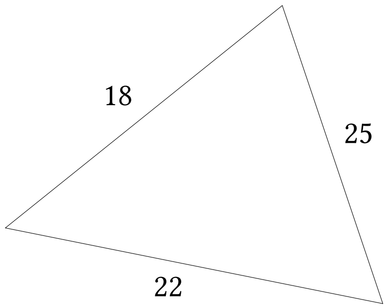


Exercises

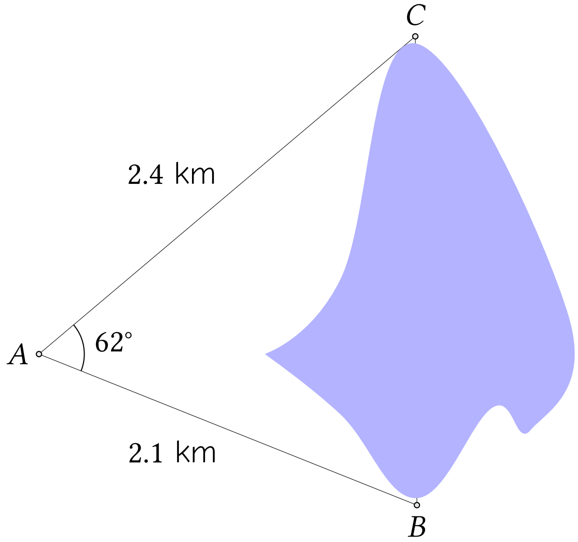
✍ **Exercise 6.2.1** Find the unknown side and angles of the triangle.



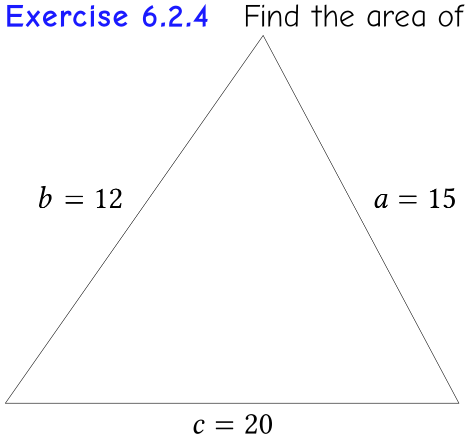
✍ **Exercise 6.2.2** Find the angles in the triangle. Round your answers to the nearest tenth.



✍ **Exercise 6.2.3** To find the distance across a small lake, a surveyor has taken the measurements shown in the figure below. Find the distance across the lake using this information. Round your answers to the nearest tenth.



✍ **Exercise 6.2.4** Find the area of the triangle in the figure below using Heron's formula.



7.1 Parabolas

Definition 7.1.1 (Geometric Definition of a Parabola)
A parabola is the set of points in the plane such that the distances $|PF$ from P to a fixed point F (called the **focus**) and the distance $|Pl|$ from P to a fixed line l (called the **directrix**) are the same.
The **axis of symmetry** is the line $l_{\perp}$ that runs through the focus and perpendicular to the directrix. The **vertex** V is the intersection of the parabola and the axis of symmetry.
The line segment that runs through the focus perpendicular to the axis, with endpoints on the parabola, is called the **latus rectum**, and its length is the **focal diameter** of the parabola.

Theorem 7.1.2 (Parabola with vertical axis) *A parabola has an equation $(x-a)^2 = 4p(y-b)$ if and only if two of the following properties are satisfied:*
(1) *the vertex is $V(h,k)$;*
(2) *the focus is $F(h,k+p)$;*
(3) *the directrix is $y = k - p$.*
The parabola opens upward if $p > 0$ or downward if $p < 0$.

Theorem 7.1.3 (Parabola with horizontal axis) *A parabola has an equation $(y-k)^2 = 4p(x-h)$ if and only if two of the following properties are satisfied:*
(1) *the vertex is $V(h,k)$;*
(2) *the focus is $F(h+p,k)$;*
(3) *the directrix is $x = h - p$.*
The parabola opens to the right if $p > 0$ or to the left if $p < 0$.

The equations in the above theorems are called the **standard form**.

Example 7.1.1 Find an equation of the parabola with the vertex $V(0,2)$ and focus $F(4,2)$.

Example 7.1.2 Find the focus, directrix, and focal diameter of the parabola $y = \frac{1}{2}x^2$.

Example 7.1.3 Find an equation of the parabola with the focus $(1, 2)$ and the directrix $y = -2$.

Example 7.1.4 A searchlight has a parabolic reflector that forms a "bowl," which is 12 in. wide from rim to rim and 8 in. deep. If the filament of the light bulb is located at the focus, how far from the vertex of the reflector is it?

Example 7.1.5 Find the vertex, focus, and directrix for the following parabola $3x - 5 = y^2 - 4y$.

Exercises

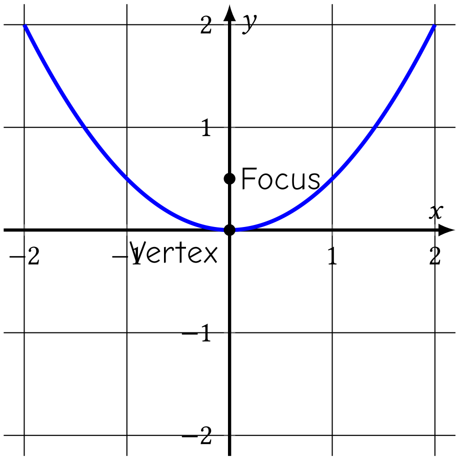
✍ **Exercise 7.1.1** Find the vertex, focus, and directrix of the parabola. Sketch the graph.
(1) $x^2 = -8(y-1)$. (2) $(y+1)^2 = 12(x-2)$. (3) $x^2 + 2x + 6y = 5$. (4) $2x - y^2 = 2$.

✍ **Exercise 7.1.2** Find an equation for the conic section with the given properties.

(1) The parabola with vertex at $(1, 0)$ and focus $(1, 5)$.

(2) The parabola with vertex at $(2, 1)$ and the directrix $x = -2$.

✍ **Exercise 7.1.3** Find an question for the conic section with the given graph.



7.2 Ellipses

Definition 7.2.1 (Geometric Definition of an Ellipse) An **ellipse** is the set of points in the plane such that, for any point P in it, the sum of distances $|PF_1|$ and PF_2 from P to two fixed points F_1 and F_2 is a constant (usually denoted by $2a$). These two fixed points are the **foci** (plural of focus) of the ellipse.

The line segment through the foci with endpoints on the ellipse is called the **major axis**

The line segment perpendicular to the major axis through the center with endpoints on the ellipse is the **minor axis**.

The intersections of the ellipse and the major axis are called the **vertices** of the ellipse. The intersections of the ellipse and the minor axis are called the **co-vertices** of the ellipse.

The midpoint of foci, or vertices, or co-vertices are the same, which is called the **center** of the ellipse.

The distance of the foci to the center is called the **focal distance** or **linear eccentricity**.

Proposition 7.2.2 *Suppose the length of the major axis is $2a$, the length of the minor axis is $2b$, and the linear eccentricity is c. Then*

$$a^2 = b^2 + c^2.$$

Theorem 7.2.3 (Ellipse with horizontal major axis) *An ellipse has an equation $\frac{(x-h)^2}{a^2} + \frac{(y-k)^2}{b^2} = 1$ if and only if two of the following properties are satisfied:*

(1) *the foci are $(h \pm \sqrt{a^2 - b^2}, k)$;*

(2) *the vertices are $(h \pm a, k)$;*

(3) *the co-vertices are $(h, k \pm b)$.*

The center of the ellipse is (h, k).

Theorem 7.2.4 (Ellipse with vertical major axis) *An ellipse has an equation $\frac{(x-h)^2}{b^2} + \frac{(y-k)^2}{a^2} = 1$ if and only if two of the following properties are satisfied:*

(1) *the foci are $(h, k \pm \sqrt{a^2 - b^2})$;*

(2) *the vertices are $(h, k \pm a)$;*

(3) *the co-vertices are $(h \pm b, k)$.*

The center of the ellipse is (h, k).

The equations of ellipses in the above theorems are called the **standard form**. The ellipses

Example 7.2.1 An ellipse has the equation $\dfrac{x^2}{9} + \dfrac{y^2}{4} = 1$. Find the foci, the vertices, and the lengths of the major and minor axes. Sketch the graph.

Example 7.2.2 Find the foci of the ellipse $16x^2 + 9(y-2)^2 = 144$.

Example 7.2.3 Find an equation of the ellipse with the vertices $(\pm 4, 1)$ and the foci $(\pm 2, 1)$.

Definition 7.2.5 The eccentricity e of an ellipse is defined as

$$e = \frac{\text{focal distance}}{\frac{1}{2}\text{(length of the major axis)}} = \frac{2 \cdot \text{(focal distance)}}{\text{(length of the major axis)}}.$$

For an ellipse centered at the origin and with the major axis along a coordinate axis, the eccentricity is $e = \dfrac{c}{a}$.

Example 7.2.4 Find the equation of the ellipse with foci $(0, \pm 8)$ and the eccentricity $e = \frac{4}{5}$.

Exercises

✍ **Exercise 7.2.1** An equation of an ellipse is given. Find the center, vertices, and foci of the ellipse, and the lengths of the major and minor axes. Sketch the graph.

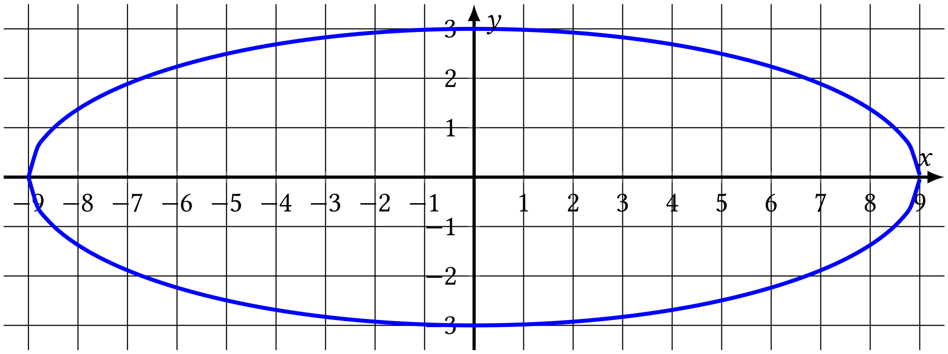
(1) $\dfrac{x^2}{9} + \dfrac{y^2}{25} = 1.$ (2) $\dfrac{(x-1)^2}{25} + \dfrac{(y+1)^2}{9} = 1.$ (3) $9x^2 + 18x + 25y^2 = -8.$

✍ **Exercise 7.2.2** Find an equation for the ellipse with the given properties.

(1) vertices $(\pm 2, 0)$ and foci $(\pm 1, 0)$.

(2) foci $(1, 4)$ and $(1, 1)$, and the eccentricity $e = \frac{3}{4}$.

✍ **Exercise 7.2.3** Find an question for the ellipse with the given graph.



7.3 Hyperbola

Definition 7.3.1 (Geometric Definition of a Hyperbola) A **hyperbola** is the set of points in the plane, such that, for any point P in it, the absolute difference of the distances $|PF_1$ and PF_2 from P to two fixed points F_1 and F_2 is a constant (usually denoted by $2a$). The fixed points F_1 and F_2 are the **foci** of the hyperbola.

The midpoint of foci is the **center** of the hyperbola.

The distance c of the foci to the center is called the **focal distance** or **linear eccentricity**.

The line segment passes through the foci and ends on the hyperbola is called the **transverse axis**.

The endpoints of the transverse axis are called the **vertices** of the hyperbola.

A hyperbola consists of two separate curves, called **branches**, that are symmetric with respect to the transverse axis, conjugate axis, and center.

Theorem 7.3.2 (Hyperbola with a horizontal transverse axis) *A hyperbola has an equation* $\frac{(x-h)^2}{a^2} - \frac{(y-k)^2}{b^2} = 1$ *if and only if the foci are* $(h \pm \sqrt{a^2+b^2}, k)$ *and the vertices are* $(h \pm a, k)$.

Theorem 7.3.3 (Hyperbola with a vertical transverse axis) *A hyperbola has an equation* $\frac{(y-k)^2}{a^2} - \frac{(x-h)^2}{b^2} = 1$ *if and only if the foci are* $(h, k \pm \sqrt{a^2+b^2})$ *and the vertices are* $(h, k \pm a)$.

Proposition 7.3.4 (Characterization of hyperbola by asymptotes) *A hyperbola centered at* (h,k) *has an equation* $\dfrac{(x-h)^2}{a^2} - \dfrac{(y-k)^2}{b^2} = r$, *where* r *is a nonzero reall number, if and only if its asymptotes are* $y - k = \pm\dfrac{b}{a}(x-h)$.

Definition 7.3.5 The rectangle whose diagonals are along the asymptotes and with a side containing a vertex of a hyperbola is called the **central box**.

The line segment through the center, perpendicular to the transverse axis with endpoints on the central box is the **conjugate axis**.

The endpoints of the conjugate axis are called the **co-vertices**.

Example 7.3.1 A hyperbola has the equation $9x^2 - 16y^2 = 121$. Find the vertices, foci, length of the transverse axis, and asymptotes. Sketch the graph.

Example 7.3.2 Find the vertices, foci, length of the transverse axis, and asymptotes of the hyperbola $x^2 + 2x - 9y^2 + 10 = 0$. Sketch the graph.

Example 7.3.3 Find the equation of the hyperbola with vertices $(\pm 3, 1)$ and foci $(\pm 4, 1)$.

Example 7.3.4 Find an equation of the hyperbola with vertices $(\pm 2, 1)$ and asymptotes $y = \pm \frac{1}{2}x + 1$.

Exercises

✍ **Exercise 7.3.1** An equation of a hyperbola is given. Find the center, vertices, foci, and asymptotes of the hyperbola. Sketch the graph.

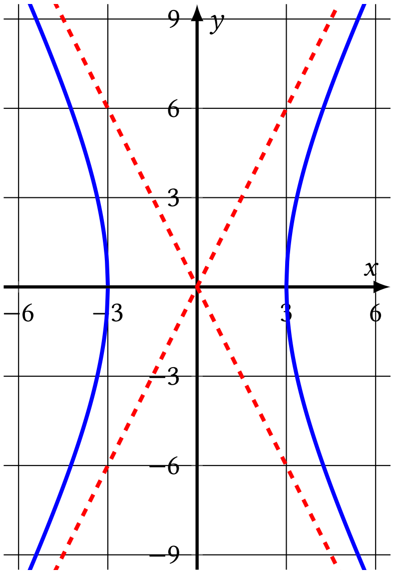
(1) $\frac{x^2}{9} - \frac{y^2}{25} = 1.$ (2) $\frac{y^2}{9} - \frac{x^2}{25} = 1.$ (3) $9x^2 - 25y^2 = 1.$ (4) $25x^2 - 9y^2 - 4 = 0.$

✍ **Exercise 7.3.2** Find an equation for the conic section with the given properties.

(1) The hyperbola with foci $(0, \pm 3)$ and vertices $(\pm 2, 0)$.

(2) The hyperbola with foci $(\pm 5, 1)$ and asymptotes $y = \pm\frac{3}{4} + 1$.

✍ **Exercise 7.3.3** Find an question for the conic section with the given graph.



Topic 8. Sequences and Series

8.1 Sequences

Definition 8.1.1 (Definition of a Sequence) A sequence is a function a whose domain is the set of natural numbers. The terms of the sequence are the function values $a(1), a(2), a(3), \cdots, a(n), \cdots$ We usually write a_n instead of the function notation $a(n)$. So the terms of the sequence are written as $a_1, a_2, a_3, \cdots, a_n, \cdots$ The number a_1 is called the first term,a_2 is called the second term,and in general,a_n is called the n-th term.

Example 8.1.1 Find the first five terms and the 100-th term of the sequence defined by each formula.

(1) $a_n = 2n - 1$ (2) $c_n = n^2 - 1$ (3) $t_n = \dfrac{n}{n+1}$ (4) $r_n = \dfrac{(-1)^n}{2^n}$

Example 8.1.2 Find the n-th term of a sequence whose first several terms are given.

(1) $\frac{1}{2}, \frac{3}{4}, \frac{4}{5}, \frac{5}{6}, \cdots$

(2) $\frac{1}{2}, \frac{3}{4}, \frac{5}{6}, \frac{7}{8}, \cdots$

(3) $-2, 4, -8, 16, \cdots$

(4) $2, 5, 8, 11, \cdots$

Definition 8.1.2 In some sequences, the n-th term may depend on some or all of the terms preceding it. Such a sequence is called **recursive sequence**.

Example 8.1.3 A sequence is defined recursively by $a_1 = 1$ and $a_n = 3(a_{n-1} + 2)$.

(1) Find the first five terms of the sequence.

(2) Find the first five terms of the sequence $\{b_n\}$ where $b_n = a_n - a_{n-1}$.

Example 8.1.4 (The Fibonacci Sequence) Find the first 11 terms of the sequence defined recursively by $F_1 = 1$, $F_2 = 1$ and

$$F_n = F_{n-1} + F_{n-2}.$$

Definition 8.1.3 (The Partial Sums of a Sequence) For the sequence $\{a_n\}$, the sum S_n for first n terms is called the n-th partial sum. The sequence $\{S_n\}$ is called the sequence of partial sums.

Example 8.1.5 Find the first four partial sums and the n-th partial sum of the sequence given by $a_n = \dfrac{1}{2^n}$.

Example 8.1.6 Find the first four partial sums and the n-th partial sum of the sequence given by $a_n = \dfrac{1}{n} - \dfrac{1}{n+1}$.

Definition 8.1.4 (Sigma notation) The sigma notation or summation notation for a partial sum of the first n-terms of a sequence a_n is defined as

$$\sum_{k=1}^{n} a_k.$$

The left side is read as "the sum of a_k from $k = 1$ to $k = n$". The letter k is called the index of summation, or the summation variable.

Example 8.1.7 Find the sum.

(1) $\sum_{k=1}^{5} k^2$

(2) $\sum_{j=3}^{5} \frac{1}{j}$

Example 8.1.8 Write each sum using sigma notation.

(1) $1^3 + 2^3 + 4^3 + \cdots + 7^3$

(2) $\sqrt{1} + \sqrt{3} + \sqrt{5} + \cdots + \sqrt{13}$

Theorem 8.1.5 (Properties of sums) *Let $\{a_n\}$ and $\{b_n\}$ be two sequences.*

(1) $\sum_{k=1}^{n} (c \cdot a_k + d \cdot b_k) = c \sum_{k=1}^{n} a_k + d \sum_{k=1}^{n} b_k$ *for any constants c and d.*

(2) $\sum_{k=1}^{n} a_k = \sum_{k=1}^{m} a_k + \sum_{k=m+1}^{n}$ *for any $1 < m < n$.*

Exercises

✍ **Exercise 8.1.1** Find the first 12-th terms of the sequence with the given n-th term.

(1) $a_n = \dfrac{n^2}{n+1}$ (2) $a_n = (-1)^n \dfrac{2^n}{n}$ (3) $a_n = \dfrac{(2n)!}{2^n n!}$

✍ **Exercise 8.1.2** Find the first nine terms of the sequence.

(1) $a_n = a_{n-1} + 2n - 1,\ a_1 = 1$

(2) $a_n = (-1)^n \dfrac{a_{n-1}}{n},\ a_1 = 1$

(3) $a_n = a_{n-1} - a_{n-2},\ a_1 = 1$ and $a_2 = 2$

✍ **Exercise 8.1.3** Find the sum.

(1) $\sum_{k=1}^{4} k^3$

(2) $\sum_{j=2}^{4} \frac{1}{j-1}$

✍ **Exercise 8.1.4** Write each sum using sigma notation.

(1) $1^3 + 3^3 + 5^3 + \cdots + 11^3$

(2) $\sqrt[3]{1} + \sqrt[3]{2} + \sqrt[3]{3} + \cdots + \sqrt[3]{10}$

✍ **Exercise 8.1.5** Find the partial sum.

(1) $\sum_{k=1}^{10} (k-1)^2$

(2) $\sum_{i=2}^{7} \frac{2i}{2i-1}$

(3) $\sum_{j=1}^{3} \frac{(-2)^j}{j+1}$

8.2 Arithmetic Sequences

Definition 8.2.1 An arithmetic sequence is a sequence of the form

$$a, a+d, a+2d, a+3d, a+4d, \cdots$$

The number a is the first term, and d is the common difference of the sequence. The n-th term of an arithmetic sequence is given by

$$a_n = a + (n-1)d.$$

Example 8.2.1 Find a_n for the arithmetic sequence

$$9, 4, -1, -6, -11, \cdots$$

Example 8.2.2 The 11-th term of an arithmetic sequence is 32, and the 19-th term is 72. Find the 100-th term.

Theorem 8.2.2 (Sum of natural numbers)

(1) $\sum\limits_{k=1}^{n} c = cn$, *where* c *is a constant.*

(2) $\sum\limits_{k=1}^{n} k = \dfrac{n(n+1)}{2}$.

Theorem 8.2.3 (Sum of an Arithmetic Sequence) *For the **arithmetic sequence** a_n, the n-th partial sum is*

$$S_n = \sum_{k=1}^{n} a_k = n\left(\frac{a_1 + a_n}{2}\right).$$

Example 8.2.3 Find the sum of the first 50 odd numbers.

Example 8.2.4 Find the following partial sum of an arithmetic sequence:

$$3 + 7 + 11 + 15 + \cdots + 159.$$

Example 8.2.5 How many terms of the arithmetic sequence $5, 7, 9, \cdots$ must be added to get 572?

Exercises

✍ **Exercise 8.2.1** Find the n-th term of the sequence and determine whether the sequence is an arithmetic sequence or not.

(1) $1-\sqrt{2},\ 1-2\sqrt{2},\ 1-3\sqrt{2},\ 1-4\sqrt{2},\ \cdots$

(2) $\sqrt{3},\ 3,\ 3\sqrt{3},\ 9,\ \cdots$

(3) $1,\ -\frac{3}{2},\ 2,\ -\frac{5}{2},\ 3,\ \cdots$

✍ **Exercise 8.2.2** Find the partial sum.

$$\frac{1}{3}+\frac{2}{3}+1+\frac{4}{3}+\frac{5}{3}+\cdots+33$$

✍ **Exercise 8.2.3** How many terms of the arithmetic sequence $3,\ 7,\ 11,\ \cdots$ must be added to get 170?

8.3 Geometric Sequences

Definition 8.3.1 (Definition of a Geometric Sequence) A **geometric sequence** is a sequence of the form

$$a, ar, ar^2, ar^3, ar^4, \cdots.$$

The number a is the first term, and r is the common ratio of the sequence. The n-th term of a geometric sequence is given by

$$a_n = ar^{n-1}.$$

Example 8.3.1 Find a_n for the geometric sequence.

(1) $2, -10, 50, -250, 1250, \cdots.$

(2) $1, \frac{1}{3}, \frac{1}{9}, \frac{1}{27}, \frac{1}{81}, \cdots.$

Example 8.3.2 The third term of a geometric sequence is $\frac{63}{4}$, and the sixth term is $\frac{1701}{32}$. Find the fifth term.

Theorem 8.3.2 (Sum of a geometric sequence) *For the geometric sequence* a, ar, ar^2, ar^3, ar^4, $\ldots$, ar^{n-1}, $ldots$, *the* n*-th partial sum is*

$$S_n = \sum_{k=1}^{n} ar^{k-1} = \frac{a(1-r^n)}{1-r}.$$

Example 8.3.3 Find the following partial sum of a geometric sequence:

$$1 + 4 + 16 + \cdots + 4096.$$

Example 8.3.4 Find the sum

$$\sum_{k=1}^{6} 7\left(-\frac{2}{3}\right)^{k-1}.$$

Definition 8.3.3 An expression of the form

$$\sum_{k=1}^{\infty} a_k = a_1 + a_2 + a_3 + a_4 + \cdots$$

is called a infinite series.

Theorem 8.3.4 (SUM OF AN INFINITE GEOMETRIC SERIES) *If $|r| < 1$, then the infinite geometric series*

$$\sum_{k=1}^{\infty} ar^{k-1} = a + ar + ar^2 + ar^3 + \cdots$$

converges and has the sum

$$S = \frac{a}{1-r}$$

If $|r| \geq 1$, the series diverges.

Example 8.3.5 Determine whether the infinite geometric series is convergent or divergent. If it is convergent, find its sum.

(1) $\sum_{k=1}^{\infty} \frac{1}{4} \cdot \left(\frac{1}{2}\right)^k$ (2) $2 + \frac{2}{5} + \frac{2}{25} + \frac{2}{125} + \cdots$ (3) $1 + \frac{7}{5} + \left(\frac{7}{5}\right)^2 + \left(\frac{7}{5}\right)^3 + \cdots$

Exercises

✍ **Exercise 8.3.1** Find the n-th term of the sequence and determine whether the sequence is a geometric sequence, or neither.

(1) $\sqrt{2},\ 2,\ 2\sqrt{2},\ 4,\ \cdots$

(2) $-1,\ \frac{4}{3},\ -\frac{5}{3},\ 2,\ \cdots$

✍ **Exercise 8.3.2** Determine whether the infinite geometric series is convergent or divergent. If it is convergent, find its sum.

(1) $1 - \frac{2}{5} + \frac{4}{25} - \frac{8}{25} + \cdots$

(2) $\frac{3}{2} + \left(\frac{3}{2}\right)^2 + \left(\frac{3}{2}\right)^3 + \cdots$

(3) $a + ab^2 + ab^3 + ab^4 + \cdots,\ |b| < 1$

(4) $a - ab^2 + ab^3 - ab^4 + \cdots,\ |b| < 1$

8.4 The Binomial Theorem

Example 8.4.1 Expand the power of binomial. Are there any relations between coefficients?
(1) $(a+b)^2$ (2) $(a+b)^3$ (3) $(a+b)^4$.

Definition 8.4.1 The product of the first n natural numbers is denoted by $n!$ and is called n factorial, that is

$$n! = n \cdot (n-1) \cdot \cdots \cdot 2 \cdot 1.$$

We define $0!$ as

$$0! = 1.$$

Definition 8.4.2 Let n and r be nonnegative integers with $r \leq n$. The binomial coefficient denoted by $\binom{n}{r}$ is defined by

$$\binom{n}{r} = \frac{n!}{r!(n-r)!} = \frac{n(n-1)\cdots(n-r+1)}{r!}.$$

Theorem 8.4.3 (Properties of binomial coefficients) *Binomial coefficients have the following properties.*

$$\binom{n}{r} = \binom{n}{n-r} = \binom{n-1}{r-1} + \binom{n-1}{r}.$$

Example 8.4.2 Calculate the binomial coefficient.
(1) $\binom{7}{3}$ (2) $\binom{50}{4}$ (3) $\binom{100}{97}$

Theorem 8.4.4 (The Binomial Theorem) *For a natural number n,*

$$(a+b)^n = \sum_{k=0}^{n} \binom{n}{k} a^{n-k} b^k,$$

where $\binom{n}{k}$ are binomial coefficients.

Example 8.4.3 Use the binomial theorem to expand $(x+y)^5$.

Example 8.4.4 Use the Binomial Theorem to expand $(\sqrt{x}-1)^6$.

Example 8.4.5 Find the term that contains x^5 in the expansion of $(2x-1)^{10}$.

Example 8.4.6 Find the term that contains x^2 in the expansion of $\left(x^3 - \frac{1}{x}\right)^{12}$.

Exercises

✍ **Exercise 8.4.1** Evaluate the expression.

(1) $\binom{5}{2}\binom{5}{3}$ (2) $\binom{5}{3}+\binom{5}{4}$ (3) $\sum_{k=0}^{5}\binom{5}{k}$ (4) $\sum_{k=0}^{8}\binom{8}{k}\binom{8}{8-k}$

✍ **Exercise 8.4.2** Expand the expression.

(1) $(2x+y)^6$ (2) $\left(x-\frac{1}{x^2}\right)^5$

✍ **Exercise 8.4.3** Find the term containing x^6 in the expansion of $(x+3)^{10}$

✍ **Exercise 8.4.4** Find the term containing no x in the expansion of $\left(4x+\frac{1}{2x}\right)^{10}$.