A Precalculus Workbook

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Preface

This book is designed to help students learn precalculus step by step and to provide instructors with a ready-to-go resource for teaching. The concise, work-along format uses an active-learning approach that builds problem-solving skills and critical thinking. It covers all essential precalculus topics: functions, polynomial and rational functions, exponential and logarithmic functions, trigonometry (including identities, the laws of sines and cosines), conic sections, sequences, and the binomial theorem.

The book works well for a one-semester precalculus course and can also serve as a review before starting calculus and is suitable as a primary text or as a supplement for active learning in class.

How Each Section is Organized

Key Definitions or Properties: Each topic begins with essential definitions and properties that form the foundation for that follows.

Learn Through Examples: Most examples are partially completed and require students to actively engage by filling in the missing steps.

Special Boxes: Different types of notes are boxed for easy identification.

- *Important Notes* (boxed with 📌) highlight information you must remember.
- *Informational Notes* (boxed with ⓘ) provide additional context or explanations.
- *Tips* (boxed with 💡) offer helpful advice or strategies for solving problems.

Key Mathematical Statements: Theorems, propositions, and results are boxed and highlighted for quick reference.

Exercises: Each section includes practice problems with answers for self-checking. Students are encouraged to attempt problems independently first.

A Note About This Book

Even with careful preparation, there will still be errors. If you find any mistakes, please let me know. Your comments, corrections, and suggestions will help make future editions better.

Acknowledgments

This project was partially supported by the CUNY OER Initiative.

Contents

Chapter 1 Introduction to Functions

1.1 Basic Concepts

Definition 1.1.1 (Basic Concepts of Functions)

A **relation** is a set of ordered pairs. The set of first components of the ordered pairs is called the **domain**, and the set of second components is called the **range**.

A **function** is a relation that assigns each element in the domain to a unique element in the range.

A value in the domain is often represented by the letter x, called an **independent variable**. A value in the range is often represented by the letter y, called a **dependent variable**. If a function has x as the independent variable and y as the dependent variable, we often say that y is a function of x. ♣

Example 1.1.1. Consider the relation

$$\{(1,2),(2,4),(3,6),(4,8),(5,10)\}.$$

1) Find the domain and the range.

2) Determine if this relation is a function.

Solution. The domain is

$$\{1,2,3,\_\_\_\_\_\_,\_\_\_\_\_\_,\_\_\_\_\_\_\}.$$

The range is

$$\{2,4,6,\_\_\_\_\_\_,\_\_\_\_\_\_,\_\_\_\_\_\_\}.$$

The relation ______ a function because for each element in the domain has a unique associated element in the range.

Example 1.1.2. Consider relation between products and prices in a grocery store.

1) Is price a function of product? If yes, what is the independent variable and what is the dependent variable?

2) Is product a function of price? If yes, what is the independent variable and what is the dependent variable?

Solution.

1) Because every product has a unique price, the price is a function of ______. The independent variable is product and the dependent variable is ______.

2) Because multiple products may have the same price, the product ________ a function of price?

Definition 1.1.2 (Function Notation)

A function is usually represented by a letter, such as f, and defined by an equation like $y = f(x)$. In this equation, $f(x)$ is called **function notation** and is read as "f of x" or "f at x." The notation $f(x)$ represents the output of the function f for a given input x. ♣

Example 1.1.3. A function $N = f(y)$ gives the number of police officers, N, in a town in year y. What does $f(2005) = 300$ represent?

Solution. From the definition of function notation, the number 2005 is the input, the number 300 is the ______. The equality means that in 2005, the number of police officers is ______.

Example 1.1.4. Consider the function $f(x) = x^2 + 3x - 4$. Find the values of the following expressions.

1) $f(2)$ 2) $f(a)$ 3) $f(a+h)$ 4) $\frac{f(a+h)-f(a)}{h}$

Solution.

1) $f(2) = 2^2 + 3(\_\_\_\_\_\_) - 4 = 6.$

2) $f(a) = \_\_\_\_\_\_^2 + 3a - 4.$

3) $f(a+h) = (\_\_\_\_\_\_)^2 + 3(\_\_\_\_\_\_) - 4 = a^2 + \_\_\_\_\_\_ + h^2 + 3a + \_\_\_\_\_\_ - 4.$

4)
$$\begin{aligned}\frac{f(a+h)-f(a)}{h} &= \frac{(a^2+2ah+h^2+3a+3h-4)-(a^2+3a-4)}{h} \\ &= \frac{\_\_\_\_\_\_ + h^2 + 3h}{h} \\ &= 2a + \_\_\_\_\_\_.\end{aligned}$$

Example 1.1.5. Consider the function $f(x) = x^2 - 2x$. Find all x values such that $f(x) = 3$.

Solution. Replacing $f(x)$ by 3 in the defining equation, we have the equation
$$x^2 - 2x = 3.$$
Solve the equation:

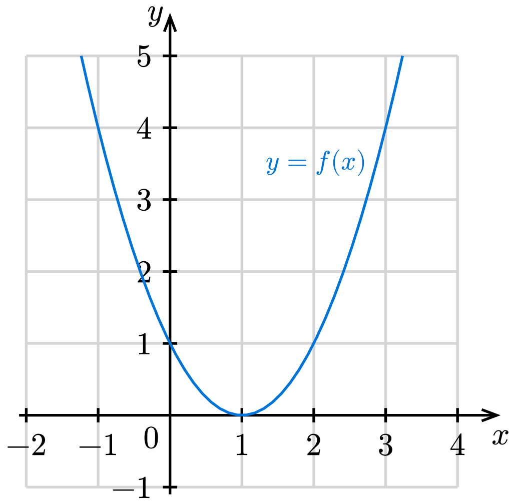
The values of x that satisfy the equation are 3 and ______.

Example 1.1.6. Express the relationship defined by the function $2x - y - 3 = 0$ as a function $y = l(x)$.

Solution. To find $l(x)$, solve the equation $2x - y - 3 = 0$ for y:
$$y = \_\_\_\_\_\_\_\_.$$
Thus, the function is $l(x) = \_\_\_\_\_\_ - 3$.

Example 1.1.7. Consider the function $f(x)$ defined by a graph below.

1) Find $f(-1)$. 2) Find all x such that $f(x) = 1$.



Solution.

1) $f(-1)$ is the y-coordinate of the point on the graph where $x = -1$. From the graph, we have
$$f(-1) = \_\_\_\_\_\_.$$

2) To find all x such that $f(x) = 3$, we look for points on the graph where the y-coordinate is 3. From the graph, we see that $f(x) = 3$ when $x = 0$ and $x =$ ______.

Definition 1.1.3 (One-to-One Function)

A function is a **one-to-one function** (also known as an **bijective function**) if every value in its range corresponds to exactly one value in the domain. ♣

Example 1.1.8. Is the function $f(x) = x^2$ one-to-one?

Solution. Because different input values, for example $x = 2$ and $x = -2$, produce the same output value $f(2) = f(-2) =$ ______, the function f ______ one-to-one.

Example 1.1.9. Is the area enclosed by a circle a function of its radius? If yes, is the function one-to-one?

Solution. The area A of a circle is given by the formula $A = \pi r^2$, where r is the radius and it is a nonnegative number. Since each radius r corresponds to exactly one area A, the area ______ a function of the radius. Moreover, the domain of the function is $r \geq 0$.

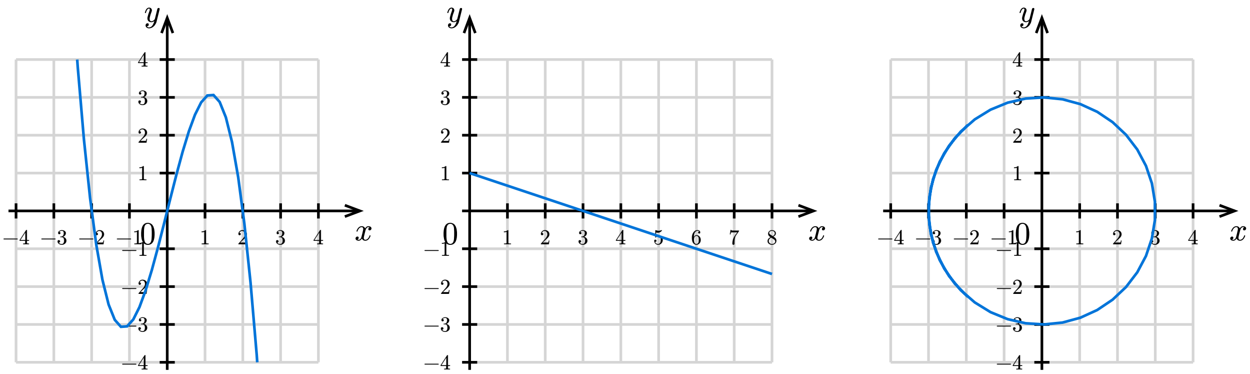
Since the radius must be nonnegative, solving r in terms of A gives a unique value of $r =$ ________ for each area A. Thus, the function is ______ one-to-one.

Horizontal and Vertical Line Test

A graph is a function if very vertical line crosses the graph at most once. This method is known as the **vertical line test**.

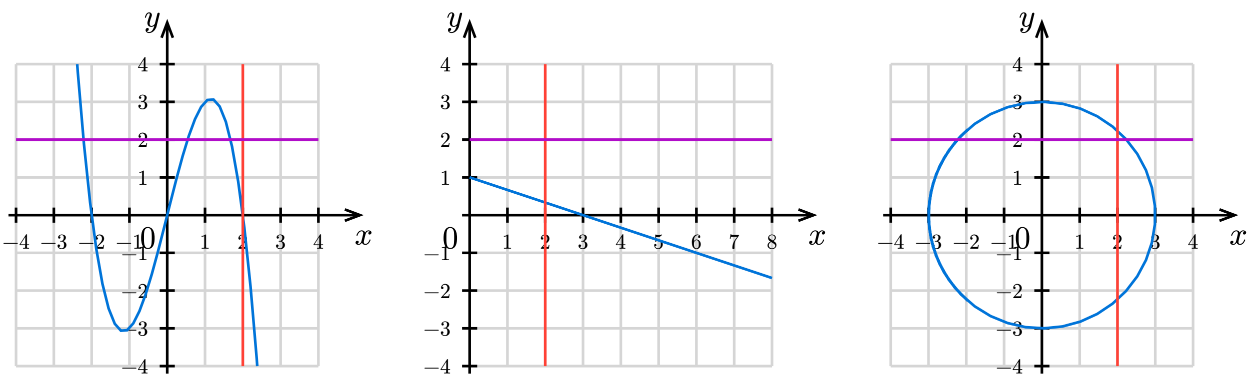
A function is an one-to-one if very horizontal line crosses the graph at most once. This method is known as the **horizontal line test**.

Example 1.1.10. Determine if the graph defines a function. If so, is it a one-to-one function?



(a) (b) (c)

Solution.



(a) (b) (c)

From the vertical line test and horizontal line test:

(a) The graph ______ a function. The function is ______ one-to-one.

(b) The graph ______ a function. The function is ______ one-to-one.

(c) The graph ______ a function. The function is ______ one-to-one.

Exercises

Exercise 1.1.1. Consider the function $f(x) = 2x^2 + x - 3$. Find the values of the following expressions.

1) $f(-1)$ 2) $f(a)$ 3) $f(a+h)$ 4) $\frac{f(a+h)-f(a)}{h}$

Answer: 1) $f(-1) = -2$ 2) $f(a) = 2a^2 + a - 3$ 3) $f(a+h) = 2a^2 + 4ah + 2h^2 + a + h - 3$ 4) $\frac{f(a+h)-f(a)}{h} = 4a + 2h + 1$

Exercise 1.1.2. Consider the function $f(x) = -x^2 - 4x$. Find all x values such that $f(x) = 3$.

Answer: The values of x that satisfy the equation are -1 and -3.

Exercise 1.1.3. Express the relationship defined by the function $3x - 2y - 6 = 0$ as a function $y = l(x)$.

Answer: $l(x) = \frac{3}{2}x - 3$.

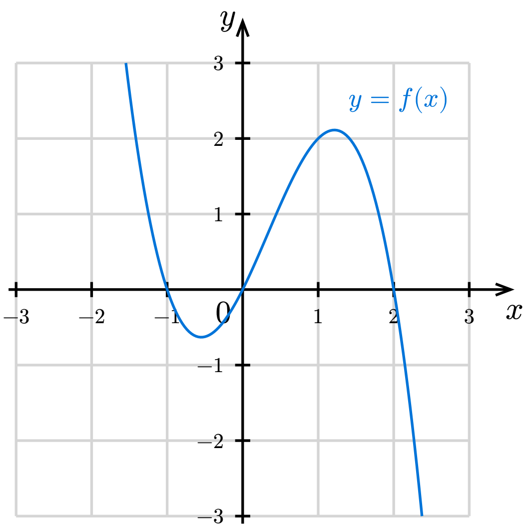
Exercise 1.1.4. Express the relationship defined by the equation $8x - y^3 = 0$ as a function $y = f(x)$. Is f a one-to-one function?

Answer: $f(x) = \sqrt[3]{8x}$. The function f is one-to-one.

Exercise 1.1.5. Consider the function $f(x)$ defined by a graph below.

1) Find $f(1)$.

2) Find all x such that $f(x) = 0$.



Answer: 1) $f(1) = 3$. 2) The values of x such that $f(x) = 0$ are -1, 0, and 2.

1.2 Domains and Ranges

Definition 1.2.1 (Domain and Range)

The **domain of a function** f consists of possible input values x. Or equivalently, the domain consists of all x values except those that will make the function is undefined.

The **range of a function** f consists of all possible output values y. Equivalently, the range consists of y value such that equation $y = f(x)$ has a solution x.

♣

Example 1.2.1. Find the domain of the function

$$f(x) = \frac{x+1}{2-x}.$$

Solution. The function is undefined when the denominator is equal to zero. So, we set the denominator equal to zero and solve for x:

$$\begin{aligned} 2 - x &= 0 \\ x &= \underline{\qquad}. \end{aligned}$$

Therefore, the domain of the function is the set of all real numbers except $x = \underline{\qquad}$.

Example 1.2.2. Find the domain of the function

$$f(x) = \sqrt{7-x}.$$

Solution. The square root $\sqrt{7-x}$ is real if the radicand $7 - x$ is nonnegative, that is $7 - x \geq 0$. Solve the inequality:

$$\begin{aligned} 7 - x &\geq 0 \\ -x &\geq \underline{\qquad} \\ x &\ \underline{\quad}\ 7. \end{aligned}$$

Therefore, the domain of the function consists of all real numbers x such that $x \leq 7$.

Set-builder and Interval Notation

Set-builder notation specifies a set of elements that satisfy a given condition. It takes the form $\{x \mid \text{statement about x}\}$, read as "the set of all x such that the statement about x is true."

Interval notation describes sets of real numbers between two endpoints, which may or may not be included. Brackets or parentheses are placed around the endpoints, separated by a comma: a square bracket indicates inclusion, and a parenthesis indicates exclusion.

For example:

- $[a, b]$ represents all real numbers from a to b, including both endpoints.
- $(a, b]$ represents all real numbers from a to b, excluding a but including b.

Example 1.2.3. Find the domain of the function $f(x) = \frac{\sqrt{x+2}}{x-1}$. Write your answer in set-builder notation and interval notation.

Solution. The function is undefined when the denominator is equal to zero or when the radicand is negative. So the domain is determined by the two conditions:

$$x - 1 \neq 0 \qquad \text{and } x + 2 \geq 0$$
$$x \neq \underline{\hspace{1.5cm}} \quad \text{and} \qquad x \geq \underline{\hspace{1.5cm}}.$$

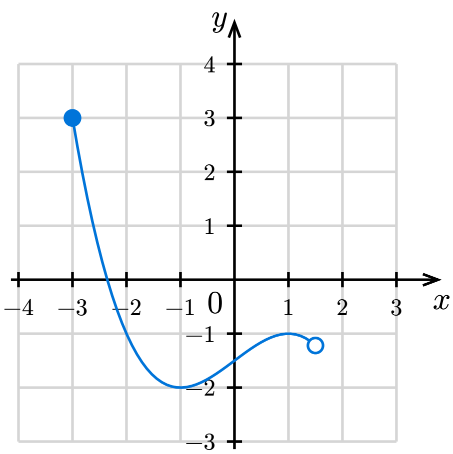
Therefore, the domain of the function in set-builder notation is

$$\{x \mid x \geq -2 \quad \text{and} \quad x \neq 1\}.$$

Since $1 \geq -2$, the domain can be expressed in interval notation as

$$[-2, \underline{\hspace{1.5cm}}) \cup (1, \underline{\hspace{1.5cm}}).$$

Example 1.2.4. Find the domain and range of the function f whose graph is shown in the following figure.



Solution. Moving a vertical line from left to right, it crosses the graph starting from $x = -3$ and up to $x = 1.5$. In interval notation, the domain is

$$[\underline{\hspace{1.5cm}}, 1.5).$$

Moving a horizontal line from bottom to top, it crosses the graph starting from $y = -2$ and up to $y = 3$. In interval notation, the range is

$$[-2, \underline{\hspace{1.5cm}}).$$

Example 1.2.5. Find the domain and range of the function

$$f(x) = 3\sqrt{x+2}.$$

Solution. The square root $\sqrt{x+2}$ is real if the radicand $x + 2$ is nonnegative, that is $x + 2 \geq 0$. Solve the inequality:

$$x + 2 \geq 0$$
$$x \geq \underline{\hspace{1.5cm}}.$$

Therefore, the domain of the function in interval is

$$[-2, \underline{\hspace{1.5cm}}).$$

When $\sqrt{x+2}$ is real, it is nonnegative, that is $\sqrt{x+2} \geq 0$. Thus,

$$f(x) = 3\sqrt{x+2} \geq 3 \cdot 0 = 0.$$

Therefore, the range of the function in interval notation is

$$[0, \underline{\hspace{1.5cm}}).$$

Definition 1.2.2 (Piecewise Function)

A **piecewise function** is a function defined by multiple sub-functions, each applying to a certain sub-interval of the main function's domain. ♣

Example 1.2.6. Consider the piecewise function

$$f(x) = \begin{cases} -2x-3 & \text{if} \quad x \le -1 \\ -x^2 & \text{if} \quad -1 < x < 1 \\ -2x+4 & \text{if} \quad 1 \le x. \end{cases}$$

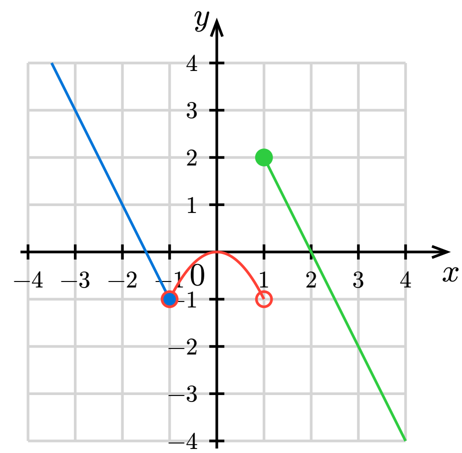
1) Sketch the graph 2) Find $f(-4)$ 3) Find $f(2)$

Solution.

1) To sketch the graph, we plot each piece of the function over its corresponding interval.

2) For $f(-4)$, since $-4 \le -1$, we use the first piece of the function:
$f(-4) = 2(-4) - 3 = -8 - 3 =$ ______.

1) For $f(2)$, since $2 \ge 1$, we use the third piece of the function:
$f(2) = -2(2) + 4 = -4 + 4 =$ ______.



Example 1.2.7. Consider the piecewise function

$$f(x) = \begin{cases} x^2-3 & \text{if} \quad x \ge -2 \\ -2 & \text{if} \quad -4 \le x < -2 \\ 5-2x & \text{if} \quad x < -4 \end{cases}$$

1) Find $f(f(-4))$ 2) Find $f\left(\frac{f(-5)}{-5}\right)$

Solution.

$$f(-4) = \_\_\_\_\_\_.$$

Next, we find $f(-2)$. Since $-2 \ge -2$, we use the first piece of the function:

$$f(-2) = (-2)^2 - 3 = \_\_\_\_\_\_.$$

$$f(-5) = 5 - 2(-5) = \_\_\_\_\_\_.$$

$$f(-3) = \_\_\_\_\_\_.$$

Exercises

Exercise 1.2.1. Find the domain of the function.

1) $f(x) = \frac{1+4x}{2x-1}$ 2) $f(x) = \sqrt{5+2x}$ 3) $f(x) = \frac{\sqrt{x+1}}{x-1}$ 4) $f(x) = \frac{x-2}{x^2+7x-44}$

Answer: **1)** $(-\infty, \frac{1}{2}) \cup (\frac{1}{2}, \infty)$. **2)** $[-\frac{5}{2}, \infty)$. **3)** $[-1, 1) \cup (1, \infty)$. **4)** $(-\infty, -11) \cup (-11, 4) \cup (4, \infty)$.

Exercise 1.2.2. Find the domain and range of each of the following functions. Write your answer in set-builder notation and interval notation.

1) $f(x) = \frac{3}{x-2}$ 2) $f(x) = -2\sqrt{x+4}$

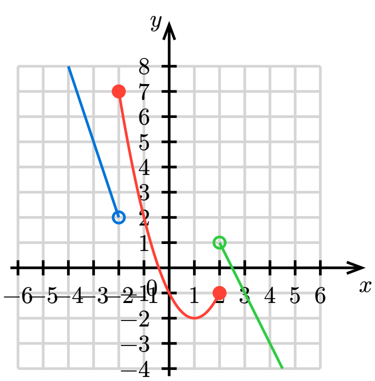
Answer: **1)** Domain: $(-\infty, 2) \cup (2, \infty)$. Range: $(-\infty, 0) \cup (0, \infty)$. **2)** Domain: $[-4, \infty]$. Range: $(-\infty, 0]$.

Exercise 1.2.3. Consider the piecewise function

$$f(x) = \begin{cases} -2x+5 & \text{if} \quad x < -2 \\ x^2 - 1 & \text{if} \quad -2 \le x \le 2 \\ 3 - 2x & \text{if} \quad 2 < x. \end{cases}$$

1) Sketch the graph. 2) Find $f(-4)$. 3) Find $f(2)$. 4) Find $f(f(3))$. 5) Find $f(f(0)+5)$.

Answer: 1)



2) $f(-3) = 5$. 3) $f(2) = -1$. 4) $f(f(3)) = 2$. 5) $f(f(0)+5) = -3$.

1.3 Monotonicity and Extrema

Definition 1.3.1 (Rate of Changes)

The **average rate of change** of a function f over an interval $[a, b]$ is defined as

$$\text{Average Rate Of Change} = \frac{f(b) - f(a)}{b - a}.$$

By taking $x = a$ and $h = b - a$, the average of rate of change is the same the **difference quotient** of a function f which is defined as

$$\text{Difference Quotient} = \frac{f(x + h) - f(x)}{h}.$$

Remark

Geometrically, the *average rate of change* is the *slope of secant line* passing through $(a, f(a))$ and $(b, f(b))$.

When h goes to 0, the *difference quotient* represents the *slope of the tangent line* passing through $(x, f(x))$.

Example 1.3.1. After picking up a friend who lives 10 miles away, Anna records her distance from home over time. The values are shown in Table. Find her average speed over the first 6 hours.

t (hours)	0	1	2	3	4	5	6	7
$D(t)$ (miles)	10	55	90	153	214	240	292	300

Solution. Anna's average speed over the first 6 hours is given by the average rate of change of $D(t)$ over the interval $[0, 6]$:

$$\frac{D(6) - D(0)}{6 - 0} = \frac{\_\_\_\_\_ - \_\_\_\_\_}{6} = \frac{290}{6} \approx \_\_\_\_\_ \text{ miles per hour.}$$

Example 1.3.2. Find the average rate of change of $f(x) = x^2 - \frac{1}{x}$ over the interval $[1, 2]$.

Solution. The average rate of change of f over the interval $[1, 2]$ is

$$\frac{f(2) - f(1)}{2 - 1} = \frac{\_\_\_\_\_ - \_\_\_\_\_}{1} = \_\_\_\_\_.$$

Example 1.3.3. Find the average rate of change of $g(t) = t^2 + 3t + 1$ on the interval $[0, a]$. The answer will be an expression involving a.

Solution. The average rate of change of g over the interval $[0, a]$ is

$$\frac{g(0) - g(a)}{0 - a} = \frac{1 - (\_\_\_\_\_)}{-a} == \frac{-(a^2 + 3a)}{-a} = \_\_\_\_\_.$$

Example 1.3.4. Find the difference quotient of $f(x) = \sqrt{x}$ at $x = a$. Make sure that the numerator is rationalized in your answer.

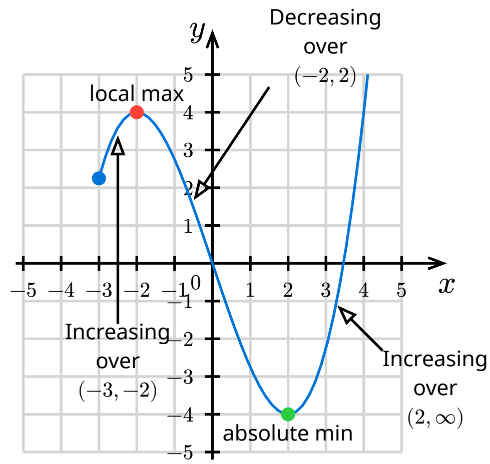
Solution. The difference quotient of f at $x = a$ is

$$\frac{f(a+h) - f(a)}{h} = \frac{\sqrt{\underline{\qquad\qquad}} - \sqrt{a}}{h}.$$

To rationalize the numerator, we multiply the numerator and denominator by the conjugate of the numerator:

$$\frac{\sqrt{a+h} - \sqrt{a}}{h} \cdot \frac{\sqrt{a+h} + \sqrt{a}}{\underline{\qquad\qquad\qquad}} = \frac{(\underline{\qquad\qquad}) - a}{h\left(\sqrt{a+h} + \sqrt{a}\right)} = \frac{h}{h\left(\sqrt{a+h} + \sqrt{a}\right)} = \underline{\qquad\qquad\qquad}.$$

Definition 1.3.2 (Monotonicity and Extrema)



A function f is **increasing** over an interval (a, b) if $f(x_2) > f(x_1)$ for any $x_1 < x_2$ in (a, b). Equivalently, f is increasing over (a, b) if the average rate of change is positive over any subinterval (x_1, x_2) of (a, b).

A function f is **decreasing** over an interval (a, b) if $f(x_2) < f(x_1)$ for any $x_1 < x_2$ in (a, b). Equivalently, f is decreasing over (a, b) if the average rate of change is negative over any subinterval (x_1, x_2) of (a, b).

A function f has a **local maximum** $f(c)$ if $f(c) \geq f(x)$ for any x near c. It has a **local minimum** $f(c)$if $f(c) \leq f(x)$ for any x near c.

A function f has an **absolute maximum** $f(c)$ if $f(c) \geq f(x)$ for all x in the domain of f. It has an **absolute minimum** $f(c)$ if $f(c) \leq f(x)$ for all x in the domain of f.

Remark

The intervals of monotonicity are usually taken as as open intervals. However, some textbooks may include the endpoints in the intervals.

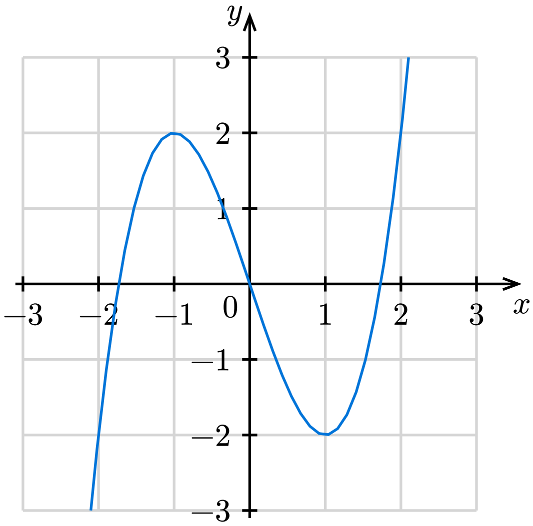
The set of points near a point $x = c$ is often called a **small neighborhood** of a point $x = c$. It is usually taken as an interval $(c - \delta, c + \delta)$ for some small positive number δ.

Theorem 1.3.3 (Local Extremum from Monotonicity)

A function f has a local maximum at $x = c$ if it switches from increasing to decreasing near c.

It has a local minimum at $x = c$ if it switches from decreasing to increasing near c.

Example 1.3.5. Find the interval of increasing and the interval of decreasing, and the local maxima and local minima of the function f defined by the following graph.



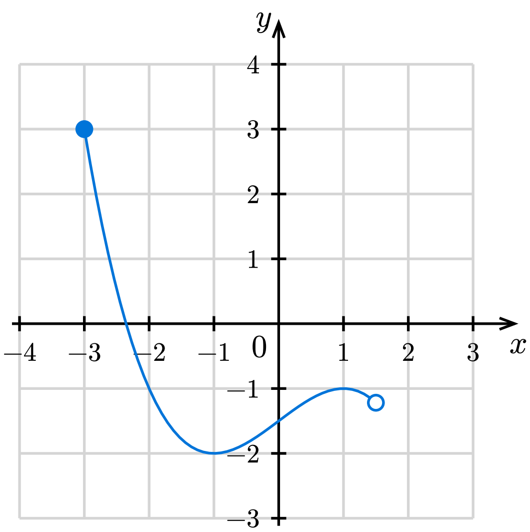
Solution. The function goes up on the left of $x = -1$ and on the right of $x = 1$. Therefore, it is increasing over the intervals $(-\infty, -1) \cup$ ________.

It goes down between $x = -1$ and $x = 1$. Therefore, it is decreasing over the interval ________.

The function switches from ________ to ________ at $x = -1$. Therefore, it has a local maximum at $x = -1$ with value $f(-1) = 2$.

It switches from ________ to ________ at $x = 1$. Therefore, it has a local minimum at $x = 1$ with value $f(1) = -2$.

Example 1.3.6. Find the local maximum, local minium, absolute maximum, and absolute minimum of the function f defined by the following graph if they exist.



Solution. The function switches from decreasing to increasing at $x = -1$. Therefore, it has a local ______ at $x = -1$ with value $f(-1) =$ ______.

Because $(-1, f(-1))$ is the lowest point on the graph, the absolute ______ is $f(-1) = -2$.

The function switches from ________ to ________ at $x = 1$. Therefore, it has a local ______ at $x = 1$ with value $f(1) =$ ______.

Because $(-3, f(-3))$ is the highest point on the graph, the absolute ______ is $f(-3) =$ ______.

Remark

Local extrema can also be found by using calculus techniques. In terms of average rate of change, a local extremum occurs where the average rate of change approaches zero as the interval shrinks to a point. It is a local maximum if the average rate of change changes from positive to negative, and it is a local minimum if the average rate of change changes from negative to positive.

Exercises

Exercise 1.3.1. The electrostatic force F, measured in newtons, between two charged particles can be related to the distance between the particles d, in centimeters, by the formula $F(d) = \frac{2}{d^2}$. Find the average rate of change of force if the distance between the particles is increased from 2 cm to 6 cm.

Answer: $-\frac{1}{9}$ N/cm".

Exercise 1.3.2. Find the average rate of change of $f(x) = x^2 + 2x - 8$ on the interval $[5, a]$.

Answer: $a + 7$.

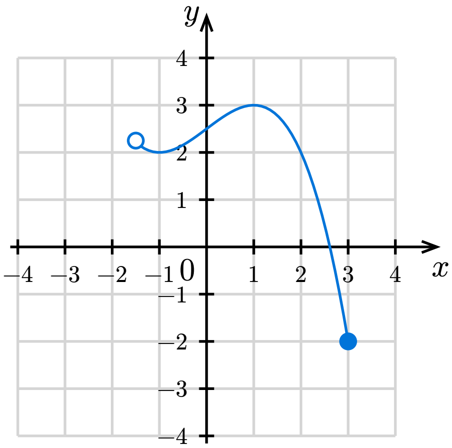
Exercise 1.3.3. Find the difference quotient of $f(x) = \sqrt{x}$ at $x = a$. Make sure that the numerator is rationalized in your answer.

Answer: $\frac{1}{\sqrt{a+h}+\sqrt{a}}$.

Exercise 1.3.4. Find the difference quotient of $f(x) = x^2 - 2x$ at $x = a$.

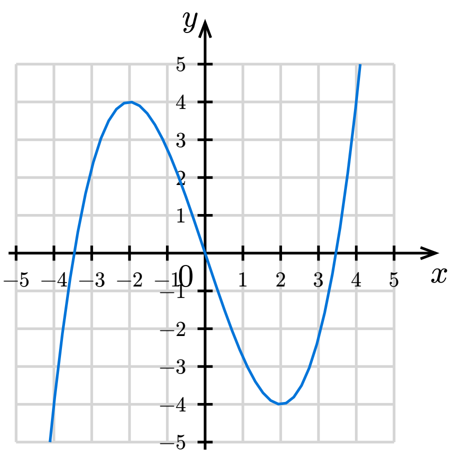
Answer: $2a - 2 + h$.

Exercise 1.3.5. Finding the absolute maximum and minimum of the function f defined by the following graph.



Answer: The absolute maximum is $f(1) = 3$. The absolute minimum is $f(3) = -2$.

Exercise 1.3.6. Find the interval of increasing and the interval of decreasing, and the local maxima and local minima of the function f using its graph.



Answer: Increasing: $(-\infty, -2) \cup (2, \infty)$; decreasing: $(-2, 2)$; local maximum: $f(-2) = 4$; local minimum: $f(2) = -4$.

1.4 Algebra of Functions

Definition 1.4.1 (Algebra of Functions)

Let f and g be two functions with domains A and B respectively. We define the linear combination, product, and quotient functions as follows.

- **Linear combination:** $(af+bg)(x)=af(x)+bg(x)$ with the domain $A\cap B$.
- **Product:** $(fg)(x)=f(x)g(x)$ with the domain $A\cap B$.
- **Quotient:** $\left(\frac{f}{g}\right)(x)=\frac{f(x)}{g(x)}$ with the domain $\{x \text{ in } A\cap B \mid g(x)\neq 0\}$.

♣

Example 1.4.1. Consider the functions $f(x)=x-1$ and $g(x)=x^2-1$.

1) Find the functions $(g-f)(x)$ and $\left(\frac{g}{f}\right)(x)$ in the simplest form.

2) Find their domains and write in interval notations.

Solution.

1) The function $(g-f)(x)$ is given by

$$(g-f)(x)=g(x)-f(x)=(\underline{\hspace{3cm}})-(\underline{\hspace{3cm}})=x^2-x.$$

The function $\left(\frac{g}{f}\right)(x)$ is given by

$$\left(\frac{g}{f}\right)(x)=\frac{g(x)}{f(x)}=\frac{x^2-1}{x-1}=\frac{(x-1)(\underline{\hspace{2cm}})}{x-1}=\underline{\hspace{2cm}},\quad x\neq 1.$$

2) Because the domains of f and g are both $(-\infty,\infty)$, the domain of $f-g$ is

$$(-\infty,\infty)\cap(-\infty,\infty)=\underline{\hspace{2cm}}.$$

The domain of $\frac{g}{f}$ is $\{x \text{ in } (-\infty,\infty)\mid f(x)\neq 0\}$. Solving $f(x)=x-1=0$ yields $x=$ ______. In interval notation, the domain of $\frac{g}{f}$ is

$$\underline{\hspace{2cm}}\cup(1,\infty).$$

Definition 1.4.2 (Compositions of Functions)

Let f and g be two functions with domains A and B respectively. The **composite function** $f\circ g$ (also called the composition of f and g) is defined as

$$(f\circ g)(x)=f(g(x)) \quad \text{with the domain} \quad \{x\in B\mid g(x) \text{ in } A\}.$$

The notation $f\circ g$ is read as "f composed with g" and means that f take g as its input. ♣

Remark

Note that in general, $f\circ g$ is not the same as $g\circ f$.

For example, let $f(x)=x+1$, $g(x)=\frac{1}{x}$, then $(f\circ g)(x)=\dfrac{1}{x}+1\neq\dfrac{1}{x+1}=(g\circ f)(x)$.

Example 1.4.2. Consider the functions $f(x) = \sqrt{x-2}$ and $g(x) = x^2 + 1$.

1) Find and simplify the functions $(f \circ g)(x)$ and $(g \circ f)(x)$. Are they the same function?
2) Find the domains of $f \circ g$ and $g \circ f$. Are they the same?

Solution.

1) The function $(f \circ g)(x)$ is given by

$$(f \circ g)(x) = f(g(x)) = f(\underline{\qquad\qquad}) = \sqrt{(x^2+1)-2} = \sqrt{\underline{\qquad\qquad}}.$$

The function $(g \circ f)(x)$ is given by

$$(g \circ f)(x) = g(f(x)) = g(\underline{\qquad\qquad}) = \left(\sqrt{x-2}\right)^2 + 1 = \underline{\qquad\qquad} + 1 = x - 1.$$

Therefore, they are not the same function.

2) The domain of f is $[2, \infty)$ and the domain of g is $(-\infty, \infty)$.

The domain of $f \circ g$ is $\{x \text{ in } (-\infty,\infty) \mid g(x) \text{ in } [2,\infty)\}$. Solving the inequality $g(x) \geq 2$ yields

$$x^2 + 1 \geq 2$$
$$x^2 \geq 1$$
$$x \geq \underline{\qquad} \text{ or } x \leq -1.$$

Thus, the domain of $f \circ g$ in interval notation is

$$(-\infty, -1] \cup \underline{\qquad\qquad}.$$

The domain of $g \circ f$ is $\{x \text{ in } [2,\infty) \mid f(x) \text{ in } (-\infty,\infty)\}$. Since $f(x)$ is always in $(-\infty, \infty)$, the domain of $g \circ f$ is the same as the domain of f, that is

$$\underline{\qquad\qquad}.$$

Example 1.4.3. Consider $f(t) = t^2 - 4t$ and $h(x) = \sqrt{x+3}$. Evaluate

1) $\frac{f(1)}{g(1)}$ 2) $(h \cdot f)(-1)$ 3) $(f \circ h)(-1)$ 4) $(3f - h)(-1)$

Solution. First find $f(1)$, $g(1)$, $f(-1)$, and $h(-1)$:

$$f(1) = (1)^2 - 4(1) = \underline{\qquad\qquad},$$
$$g(1) = \sqrt{1+3} = \underline{\qquad\qquad},$$
$$f(-1) = (-1)^2 - 4(-1) = \underline{\qquad\qquad},$$
$$h(-1) = \sqrt{-1+3} = \underline{\qquad\qquad}.$$

1) $\dfrac{f(1)}{g(1)} = \dfrac{\underline{\qquad\qquad}}{\underline{\qquad\qquad}} = \underline{\qquad}.$
2) $(h \cdot f)(-1) = h(f(-1)) = h(\underline{\qquad\qquad}) = \sqrt{\underline{\qquad\qquad}} = \underline{\qquad}.$
3) $(f \circ h)(-1) = f(h(-1)) = f(\underline{\qquad\qquad}) = (\underline{\qquad\qquad})^2 - 4(\underline{\qquad\qquad}) = \underline{\qquad}.$
4) $(f - h)(-1) = 3f(-1) - h(-1) = 3(\underline{\qquad\qquad}) - \underline{\qquad\qquad} = \underline{\qquad}.$

Example 1.4.4. Using the graphs to evaluate the given functions.

1) $(f+g)(1)$

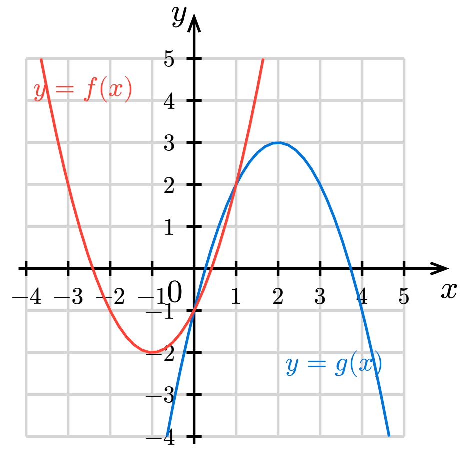
2) $(fg)(1)$

3) $\left(\frac{f}{g}\right)(1)$

4) $f^2(1) - \left(\frac{3g}{f}\right)(0)$

5) $(g \circ f)(-3)$

6) $(f \circ g)(0))$



Solution. First find the values of $f(1)$, $g(1)$, $f(-3)$, $f(0)$, and $g(0)$ from the graph:
$f(1) =$ ______, $g(1) =$ ______, $f(-3) =$ ______, $f(0) =$ ______ $g(0) =$ ______.
Therefore,

1) $(f+g)(1) = f(1) + g(1) =$ ______________.

2) $(fg)(1) = f(1)g(1) =$ ______________.

3) $\left(\frac{f}{g}\right)(1) = \frac{f(1)}{g(1)} =$ ______________.

4) $f^2(1) - \left(\frac{3g}{f}\right)(0) = (f(1))^2 - \frac{3g(0)}{f(0)} =$ ______________.

5) $(g \circ f)(-3) = g(f(-3)) = g($______$) =$ ______________.

6) $(f \circ g)(0) = f(g(0)) = f($______$) =$ ______________.

Example 1.4.5. Consider the function $h(x) = \sqrt{x^2+1}$. Find two non-identity functions f and g so that $h(x) = f(g(x))$.

Solution. One possible answer is $f(x) = \sqrt{x}$ and $g(x) =$ __________.

Exercises

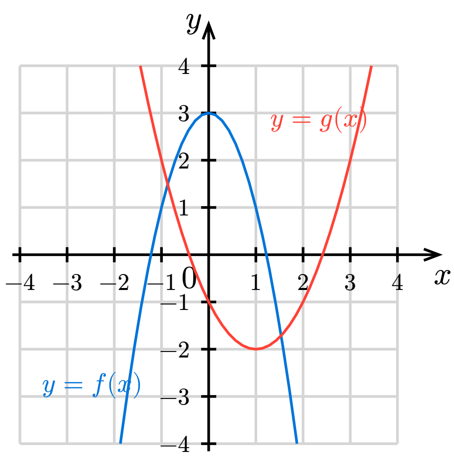
Exercise 1.4.1. Consider the functions $f(x) = x^2 - 1$ and $g(x) = x + 1$.

1) Find the functions $f - g$ and $\frac{f}{g}$, and their domains.
2) Find $(f^2 - 3g)(1)$.
3) Find $\left(2fg - \frac{3f}{g}\right)(2)$.

Answer: 1) $(f-g)(x) = x^2 - x - 2$, domain: $(-\infty, \infty)$; $\left(\frac{f}{g}\right)(x) = x - 1$, domain: $(-\infty, 1) \cup (1, \infty)$. 2) -6. 3) 15.

Exercise 1.4.2. Using the graphs to evaluate the given functions.

1) $(f + g)(1)$
2) $(fg)(1)$
3) $\left(\frac{f}{g}\right)(1)$
4) $(g \circ f)(-1)$
5) $f(g(0))$
6) $g(10 - f^2(0))$
7) $\left(\frac{f}{g}\right)(g(0))$



Answer: 1) 2. 2) 1. 3) 1. 4) 1. 5) 1. 6) -2. 7) $\frac{1}{2}$

Exercise 1.4.3. Consider the functions $f(x) = \frac{1}{x-2}$ and $g(x) = \sqrt{x+4}$.

1) Find $f \circ g$ and its domain.

2) Find $(g \circ f)(3)$.

Answer: 1) $(f \circ g)(x) = \frac{1}{\sqrt{x+4}-2}$, domain: $(-4, 0) \cup (0, \infty)$. 2) $\sqrt{5}$.

Exercise 1.4.4. Consider the function $h(x) = \sqrt[3]{2x-1}$. Find two non-identity functions f and g so that $h(x) = f(g(x))$.

Answer: $f(x) = \sqrt[3]{x}$, $g(x) = 2x - 1$.

1.5 Transformations

Definition 1.5.1 (Shifting)

Let f and g be two functions, and C and D be two real numbers.

If $g(x) = f(x) + D$, then the graph of g is obtained by shifting the graph of f by D units. We call the transformation from f to g a **vertical shift** by D units.

If $g(x) = f(x - C)$, then the graph of g is obtained by shifting the graph of f by C units. We call the transformation from f to g a **horizontal shift** by C units. ♣

Direction of Shift

The signs of C and D determine the direction of the shift. A *positive* sign indicates an *upward* or *rightward* shift. A *negative* sign indicates a *downward* or *leftward* shift.

Example 1.5.1. The point $(9, -15)$ is on the graph of $y = f(x)$.

1) Find a point on the graph of $g(x) = f(x) + 5$.
2) Find a point on the graph of $g(x) = f(x + 5)$.

Solution.

1) Because $f(9) = -15$,
$$g(9) = f(\_\_\_\_\_\_) + 5 = \_\_\_\_\_\_.$$
Therefore, $(9, \_\_\_\_\_\_)$ is a point on the graph of $g(x) = f(x) + 5$.
2) Because $f(9) = -15$ and $g(x) = f(x + 5)$, if we let x be the solution of $x + 5 = 9$, that is, $x = \_\_\_\_\_\_$, then
$$g(4) = f(\_\_\_\_\_\_ + 5) = f(9) = -15.$$
Therefore, $(4, -15)$ is a point on the graph of $g(x) = f(x + 5)$

Example 1.5.2. Consider the functions $f(x) = x^2$, $g(x) = x^2 - 1$ and $h(x) = x^2 + 2$.

1) Describe how to get the graph of g from the graph of f.
2) Describe how to get the graph of h from the graph of g.
3) Describe how to get the graph of f from the graph of h.

Solution. To determine the shift from f to g, write the function g as a function of f and find the units of shift.

1) Because $g(x) = f(x) + 1$, the graph g is a upward shift of the graph of f by 1 unit.
2) Because $h(x) = g(x) + \_\_\_\_\_\_$, the graph h is a upward shift of the graph g by 3 units.
3) Because $f(x) = h(x) + (-2)$, the graph of f is a _______ shift of the graph h by 2 units.

Example 1.5.3. Consider the functions $f(x) = x^2$, $g(x) = (x+1)^2$ and $h(x) = (x-2)^2$.

1) Describe how to get the graph of g from the graph of f.
2) Describe how to get the graph of h from the graph of g.
3) Describe how to get the graph of f from the graph of h.

Solution. To determine the transformation from f to g, write the function g as a function of f and find the units of shift.

1) Because $g(x) = f(x+1) = f(x-(-1))$, the graph g is a shift of the graph of f to the ________ by 1 unit.
2) Because $h(x) = g(x-3)$, the graph h is a shift of the graph g to the ________ by 3 units.
3) Because $f(x) = h(x+2) = h(x-(\_\_\_\_\_\_))$, the graph of f is a shift of the graph h to the ________ by 2 units.

How to Find the Horizontal Shift C

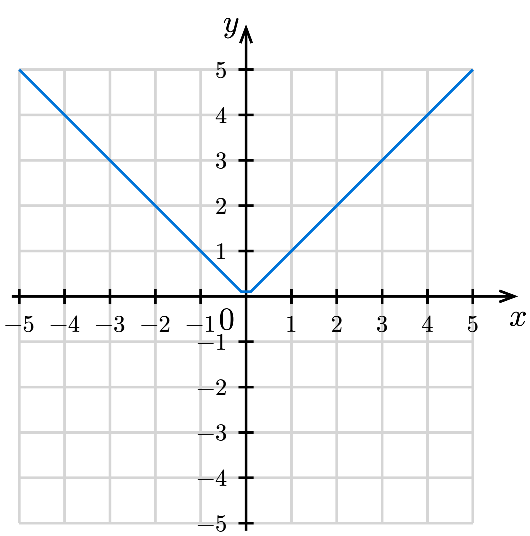
Suppose that $g(x) = f(x+k)$. The horizontal shift C can be found by solving the equation $C + k = 0$. Thus, $C = -k$.

Example 1.5.4. Sketch the graph of $f(x) = |x|$. Then use the graph to sketch the graph of $h(x) = f(x+2) - 1$.

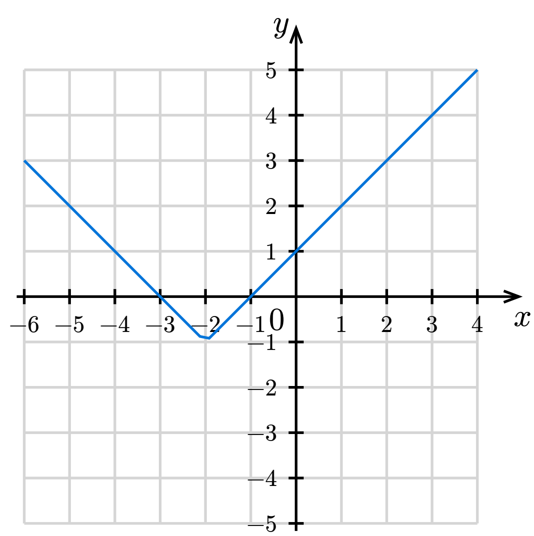
Solution. By the definition of absolute value, the function $f(x) = |x|$ can be written as

$$f(x) = \begin{cases} x & x \geq 0 \\ -x & x < 0 \end{cases}.$$

Since $D = -1$, and $C = -2$, the graph of h is a shift of the graph of f to the left by 2 units and downward by 1 unit. The graphs of f and h are shown below.

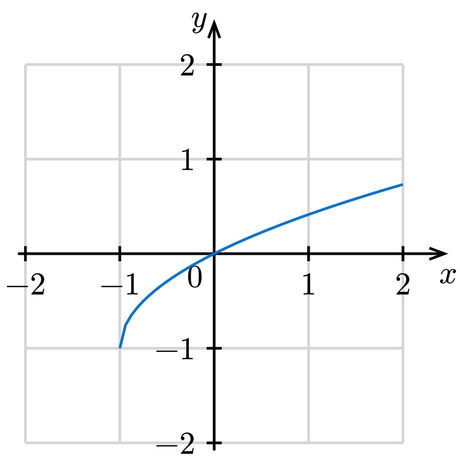


$y = f(x) = |x|$



$h(x) = f(x+2) - 1 = |x+2| - 1$

Example 1.5.5. The function $y = g(x)$ shown in the picture is a shift of the square root function $y = \sqrt{x}$. Find $g(x)$.



Solution. Since the graph of g is a shift of the graph of $f(x) = \sqrt{x}$, the function g is defined by $g(x) = f(x - C) + D$. Note that the function f has a starting point at $(0, 0)$, and the function g has a starting point at $(-1, -1)$. The defining equation of g implies the two points are related by the the system of equations:

$$-1 = 0 - C \qquad\qquad -1 = 0 + D$$
$$C = 1, \qquad\qquad D = -1.$$

Thus, the function g is given by $g(x) = f(x - 1) - 1 = \sqrt{(x-1)+1} - 1 = \sqrt{x} - 1$.

Definition 1.5.2 (Scaling)

Let f and g be two functions, and C and D be two real numbers. Assume that $A > 0$ and $B > 0$.

If $g(x) = Af(x)$, then the graph of g is obtained by scaling the graph of f by a factor of A in the vertical direction. We say that the transformation from f to g a **vertical scaling** of the function $y = f(x)$ by a factor of A.

If $g(x) = f(Bx)$, then the graph of g is obtained by scaling the graph of f by a factor of $\frac{1}{B}$ in the horizontal direction. We say that the transformation from f to g a **horizontal scaling** of the function $y = f(x)$ by a factor of $\frac{1}{B}$.

Remark

If the factor A or $\dfrac{1}{B}$ is greater than 1, the scaling is a stretch, and it is between 0 and 1, the scaling is a compression.

If $A < 0$ or $B < 0$, then the function g is obtained from f by a reflection (see Definition Definition 1.5.3) and a scaling.

Example 1.5.6. The point $(9, -15)$ is on the graph of $y = f(x)$.

1) Find a point on the graph of $g(x) = \frac{1}{3}f(x)$.
2) Find a point on the graph of $g(x) = f(3x)$

Solution.

1) Because $f(9) = -15$,

$$g(9) = \frac{1}{3}f(9) = \frac{1}{3}(\underline{\hspace{2cm}}) = \underline{\hspace{2cm}}.$$

Therefore, $(9, \underline{\hspace{2cm}})$ is a point on the graph of $g(x) = \frac{1}{3}f(x)$.

2) Because $f(9) = -15$ and $g(x) = f(3x)$, if we let x be the solution of $3x = 9$, that is, $x = \underline{\hspace{2cm}}$, then

$$g(3) = f(3 \cdot (\underline{\hspace{2cm}})) = f(9) = -15.$$

Therefore, $(\underline{\hspace{2cm}}, -15)$ is a point on the graph of $g(x) = f(3x)$.

Example 1.5.7. Describe how to get the graph of the function $g(x) = 4x^2$ from the graph of the function $f(x)$.

Solution.

Option 1: Since $g(x) = 4f(x)$, the graph of g can be obtained from the graph of $f(x) = x^2$ by a vertical stretch by a factor of 4.

Option 2: Since $g(x) = f(2x)$, the graph of g can be obtained from the graph of $f(x) = x^2$ by a horizontal compression by a factor of $\frac{1}{2}$.

Definition 1.5.3 (Reflections)

Let f and g be two functions.

If $g(x) = -f(x)$, then the graph of g is obtained by reflecting the graph of f about the x-axis. We say that the transformation from f to g a **vertical reflection** of the function $y = f(x)$.

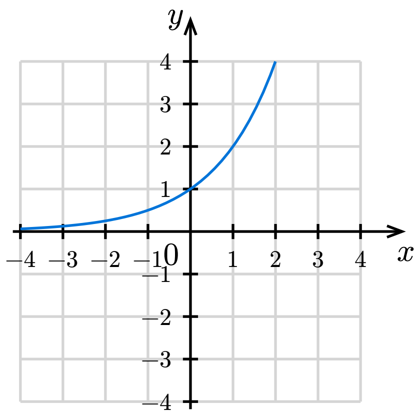
If $g(x) = f(-x)$, then the graph of g is obtained by reflecting the graph of f about the y-axis. We say that the transformation from f to g a **horizontal reflection** of the function $y = f(x)$.

Example 1.5.8. Reflect the graph of $f(x) = |x - 1|$ about the x-axis and then about the y-axis. Denote the resulting function by $y = g(x)$. Find a defining equation of g.

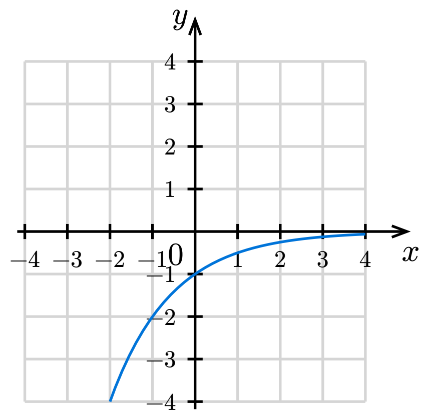
Solution. Reflecting the graph of f about the x-axis gives the function $y = -f(x) = -|x - 1|$. Reflecting the graph of $y = -f(x)$ about the y-axis gives the function

$$y = g(x) = -f(-x) = \underline{\hspace{3cm}} = -|x + 1|.$$

Example 1.5.9. The graph of the function $f(x) = 2^x$ is shown below. Use reflections to sketch the graph of the function $g(x) = -\left(\frac{1}{2}\right)^x$.



$f(x) = 2^x$



$g(x) = -\left(\frac{1}{2}\right)^x$

Solution. The function $g(x) = -\left(\frac{1}{2}\right)^x$ can be written as $g(x) = -f(-x)$, where $f(x) = 2^x$. Thus, the graph of g can be obtained from the graph of f by reflecting the graph of f about the y-axis and then about the x-axis. The graph of g is shown below.

Definition 1.5.4 (Even and Odd Functions)

A function f is an **even function** if $f(-x) = f(x)$ for all x in the domain of f.

A function f is an **odd function** if $f(-x) = -f(x)$ for all x in the domain of f. ♣

Symmetry of Even and Odd Functions

The graph of an even function is **symmetric about y-axis**, that is, if (x, y) is on the graph, then $(-x, y)$ is also on the graph.

The graph of an odd function is **symmetric about the origin**, that is, if (x, y) is the on the graph, then $(-x, -y)$ is also on the graph.

Example 1.5.10. Determine whether the functions is even, odd, or neither.

1) $f(x) = x^2 - 1$ 2) $g(x) = |x - 1|$ 3) $h(x) = x^3 - 2x$ 4) $k(x) = \frac{1}{x^2}$.

Solution.

1) Because $f(-x) = (-x)^2 - 1 = x^2 - 1 = f(x)$, the function f is an even function.
2) Because $g(-x) = |-x - 1| = |x + 1| \neq |x - 1| = g(x)$ and $g(-x) \neq -g(x)$, the function g is neither even nor odd.
3) Because $h(-x) = (-x)^3 - 2(-x) = -x^3 + 2x = -(x^3 - 2x) = -h(x)$, the function h is an odd function.
4) Because $k(-x) = \frac{1}{(-x)^2} = \frac{1}{x^2} = k(x)$, the function k is an even function.

Order of Transformations

To get the graph of the function $g(x) = Af(Bx + C) + D$ from the graph of the function $y = f(x)$, the order of horizontal or vertical transformations depends on how to get a point (x, y) on the graph g from a point (a, b) on the graph of f. If (a, b) is a point on the graph of f, then *the solution* (x, y) of the system of linear equations

$$\begin{cases} a = Bx + C \\ y = Ab + D \end{cases}$$

is *the point on the graph of* g obtained by transformations from the point (a, b). The order of transformations depends on the order of algebraic operations used to solve for x and y.

One possible order of transformations is as follows:

- **Vertical transformations** (from the left (A) to the right (D)):
  1) A vertical stretch/compression with the factor $|A|$
  2) A vertical refection if $A < 0$.
  3) A vertical shift of D units
- **Horizontal transformations** (from the right (C) to the left (B)):
  1) A horizontal shift of $-C$ units.
  2) A horizontal stretch/compression with the factor $\frac{1}{|B|}$.
  3) A horizontal refection about y-axis if $B < 0$.

The two groups of transformations can be switched as x and y can be solved individually.

Relection and scaling can be switced because of the commutativity of multiplication.

However, the order of shift determines the units of shift.

Example 1.5.11. Describe how to obtain the graph $g(x) = -2f(3x - 6) + 4$ from the graph of the function f.

Solution. To get the graph of the function g from the graph of the function f, we can perform the vertical transformations first, followed by the horizontal transformations.

- For vertical transformations, working with from left to right with A first and then D:
  1) A vertical stretch by a factor of 2.
  2) A reflection about the x-axis.
  3) A vertical shift upward by 4 units.
- Horizontal transformations correspond to how $Bx + C = 0$ is solved:

$$\begin{aligned} 3x - 6 &= 0 \\ 3x - 6 + 6 &= 0 + 6 \implies \\ 3x &= 6 \\ \tfrac{1}{3}\cdot(3x) &= \tfrac{1}{3}\cdot 2 \implies \\ x &= \tfrac{2}{3}. \end{aligned}$$

  4) A horizontal shift of 6 units to ________.
  5) A horizontal compression by a factor of ______.

Example 1.5.12. Find an equation of the function $y = g(x)$ whose graph is obtained from $f(x) = \sqrt{x}$ by the following transformations in the given order.

1) Stretch vertically by a factor of 2.
2) Shift downward 2 units.
3) Shift 3 units to the left.
4) Stretch horizontally by a factor $\frac{1}{2}$.

Solution. Let $f(x) = \sqrt{x}$. The graph of g can be obtained from the graph of f by the following transformations:

1) A vertical stretch by a factor of 2: $g_1(x) =$ ______ $f(x) = 2\sqrt{x}$.
2) A vertical shift downward by 2 units: $g_2(x) = g_1(x) +$ ______ $= 2\sqrt{x} - 2$.
3) A horizontal shift to the left by 3 units: $g_3(x) = g_2(x +$ ______$) = 2\sqrt{x+3} - 2$.
4) A horizontal stretch by a factor of $\frac{1}{2}$: $g(x) = g_3($______ $x) = 2\sqrt{\frac{1}{2}x + 3} - 2$.

Therefore, the equation of the function g is given by

$$g(x) = 2\sqrt{\frac{1}{2}x + 3} - 2.$$

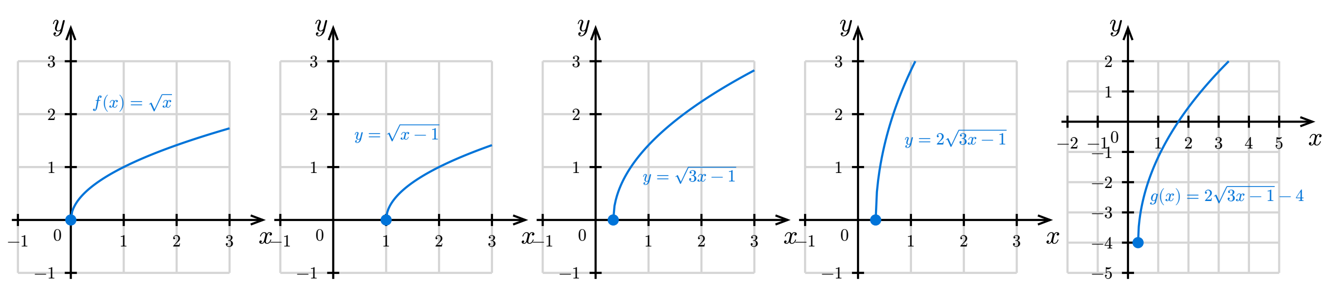
Remark

When shifting horizontally, replace x with $x - C$, where $|C|$ is the number of units shifted. The sign of C matches the direction: right means positive, left means negative.

Example 1.5.13. Sketch the graph of the function $g(x) = 2\sqrt{3x-1} - 4$ by a sequence of transformation applied on the graph of $f(x) = \sqrt{x}$.

Solution. The graph of g can be obtained from the graph of $f(x) = \sqrt{x}$ by the following transformations:

1) A horizontal shift to ________ by 1 unit.
2) A horizontal compression by a factor of ______.
3) A vertical stretch by a factor of ______.
4) A vertical shift ______ by 4 units.



Exercises

Exercise 1.5.1. Consider the functions $f(x) = x^2$, $g(x) = (x+1)^2 - 2$ and $h(x) = (x-2)^2 + 1$.

1) Describe how to get the graph of g from the graph of f.

2) Describe how to get the graph of h from the graph of g.

Answer:
1) Shift the graph of f to the left by 1 unit and downward by 2 units.
2) Shift the graph of g to the right by 3 units and upward by 3 units.

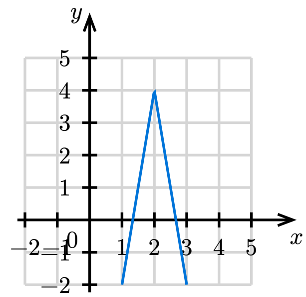
Exercise 1.5.2. Determine whether the function is even, odd, or neither.

1) $f(x) = 1 - x^2$ 2) $g(x) = \sqrt[3]{-x}$ 3) $h(x) = x^4 - x^3$

Answer: 1) The function f is even. 2) The function g is odd. 3) The function h is neither even nor odd.

Exercise 1.5.3. Sketch the graph of the function $g(x) = -2|3x - 6| + 4$ by a sequence of transformation applied on the graph of $f(x) = |x|$.

Answer:



Exercise 1.5.4. Find an equation of the function $y = g(x)$ whose graph is obtained from $f(x) = \sqrt[3]{x}$ by the following transformations in the given order.

1) Compress vertically by a factor of $\frac{1}{2}$.

2) Reflect vertically.

3) Shift downward 2 units.

4) Compress horizontal by a factor 2.

5) Shift 3 units to the right.

Answer: $g(x) = -\frac{1}{2}\sqrt[3]{\frac{1}{2}x - 3} - 2$.

Exercise 1.5.5 (Optional). Describe how to get $f(x) = \sqrt{x}$ from $g(t) = -\frac{1}{2}\sqrt{2t+1} - 3$. (Hint: find a defining equation f using g.)

Answer:

1) Shift the graph of g upward by 3 units.
2) Reflect the graph of g about the x-axis.
3) Stretch the graph of g vertically by a factor of 2.
4) Stretch the graph of g horizontally by a factor of $\frac{1}{2}$.
5) Shift the graph of g to the left by 1 unit.

1.6 Inverse Functions

Definition 1.6.1 (Inverse Functions)

Let $y = f(x)$ be a function with the domain A. A function $f^{-1}(x)$ with the domain B is an **inverse function** of f if $f^{-1}(f(x)) = x$ for all in B and $f(f^{-1}(x))$ for all x in A.

The notation f^{-1} is read as "f inverse."

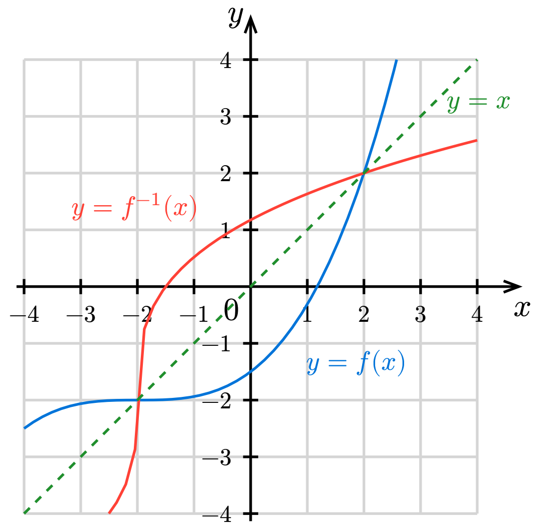
Properties of Inverse Functions

1) If a function f has an inverse function, then it has a unique inverse function.
   Proof: Suppose g is also an inverse f. Then $f(g(x)) = x = f(f^{-1}(x))$. Then $g(x) = f^{-1}(f(g(x))) = f^{-1}(f(f^{-1}(x))) = f^{-1}(x)$.

2) Note that if f^{-1} is the inverse of f, then f is also the inverse of f^{-1} that is $f(f^{-1}(x)) = x$ for all x in the domain of f^{-1}.

3) In general, $f^{-1}(x) \neq f(x)^{-1}$. For example, if $f(x) = 2x$, then $f^{-1}(x) = \frac{x}{2}$, but $f(x)^{-1} = \frac{1}{2x}$.

4) The graphs of a one-to-one function f and its inverse f^{-1} are symmetric about the diagonal line $y = x$.



5) Suppose f has the domain A and the range B, then f^{-1} has the domain B and the range A (and vice verse).

6) If f is a one-to-one (bijective) function, then f has an inverse function.

Remark

If g is a function such that $f(g(x)) = x$, then g is called a **right inverse**. If $g(f(x)) = x$, then g is called a **left inverse**. If f has a left inverse, then $a = b$ if $f(a) = f(b)$ and f is called an **injective** function. If f has right inverse, then for any y in the range of f, there is an $x = g(y)$ in the domain of f such that $f(x) = y$ and f is called a **surjective** (or onto) function.

Example 1.6.1. Let f be a one-to-one function with $f(3) = 4$ and $f(4) = 5$. Find $f^{-1}(4)$.

Solution.

Because $f(3) = 4$, we have

$$f^{-1}(4) = f^{-1}(f(3)) = \underline{\qquad}.$$

Example 1.6.2. Let $f(x) = \frac{1}{x-1}$ and $g(x) = \frac{x+1}{x}$. Determine if g is the inverse function of f.

Solution. Find $f(g(x))$ and $g(f(x))$.

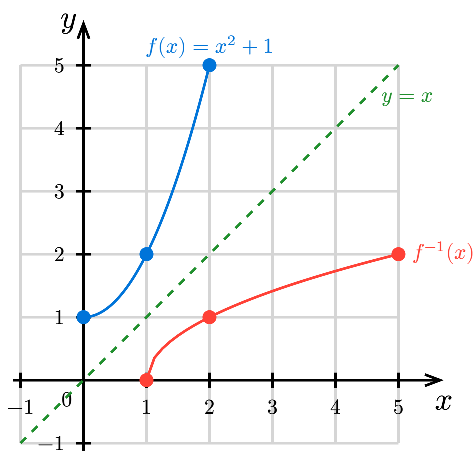
$$f(g(x)) = f\left(\frac{x+1}{x}\right) = \frac{1}{\frac{x+1}{x} - 1} = \frac{1}{\underline{\qquad}} = x.$$

$$g(f(x)) = g\left(\frac{1}{x-1}\right) = \frac{\frac{1}{x-1} + 1}{\frac{1}{x-1}} = \frac{\underline{\qquad}}{\frac{1}{x-1}} = x.$$

Because $f(g(x)) = x$ and $g(f(x)) = x$, the function g is the inverse function of f.

Example 1.6.3. Consider the function $f(x) = x^2 + 1$ with $x > 0$. Sketch the graph of $y = f^{-1}(x)$ without finding its equation.

Solution. The graph of f^{-1} is the reflection of the graph of f about the line $y = x$. To sketch the graph of f^{-1}, we can plot some points on the graph of f and then reflect them about the line $y = x$.



Existance of Inverse Functions

Given a function $y = f(x)$, if f is one-to-one (bijective), then the ***inverse function*** is the ***unique solution*** y ***of the equation*** $f(y) = x$.

If f is not one-to-one, then the inverse function may not exist over its full domain. However, an inverse function can exist by restricting f to a subdomain where it is one-to-one.

For example, the function $f(x) = x^2$ is not one-to-one on $(-\infty, \infty)$, but if we restrict the domain to $[0, \infty)$, the inverse function is $f^{-1}(x) = \sqrt{x}$.

Example 1.6.4. Consider the function $f(x) = 2x - 3$. Find the inverse function f^{-1} and its domain and range.

Solution. To find the inverse function f^{-1}, we solve for y from the equation $f(y) = x$:

$$\begin{aligned} 2y - 3 &= x \\ 2y &= \_\_\_\_\_\_ \\ y &= \_\_\_\_\_\_ \ . \end{aligned}$$

Therefore, the inverse function is given by

$$f^{-1}(x) = \frac{x+3}{2}.$$

The domain and range of f are both $(-\infty, \infty)$. Thus, the domain and range of f^{-1} are also both ________.

Example 1.6.5. Consider the function $f(x) = \frac{x}{x-1}$.

1) Find the inverse function f^{-1} and its domain and range.

2) Find the range of f.

Solution.

1) To find the inverse function f^{-1}, we solve for y from the equation $f(y) = x$:

$$\begin{aligned} \frac{y}{y-1} &= x \\ y &= x(y-1) \\ y &= xy - x \\ \_\_\_\_\_\_ &= -x \\ y(1-x) &= -x \\ y &= \_\_\_\_\_\_ \ . \end{aligned}$$

Therefore, the inverse function is given by

$$f^{-1}(x) = \frac{-x}{1-x}.$$

Because $f(x)$ is also a rational expression, the domain of f can be found similary and it is

$$(-\infty, 1) \cap \_\_\_\_\_\_\_\_.$$

Thus, the range of f^{-1} is ________.

Since $f^{-1}(x)$ is a rational expression, it is undefined if $1 - x = 0$, equivalently, $x =$ ______. Thus, in interval notation, the domain of f^{-1} is

$$\_\_\_\_\_\_\_\_ \cap (1, \infty).$$

2) The range of f is the domain of f^{-1}, that is,

$$\_\_\_\_\_\_\_\_.$$

Example 1.6.6. Consider the function $f(x) = \sqrt{x-2}$. Find the inverse function f^{-1} and its domain and range.

Solution. To find the inverse function f^{-1}, we solve for y from the equation $f(y) = x$:

$$\begin{aligned} \sqrt{y-2} &= x \\ y-2 &= x^2 \\ y &= x^2+2. \end{aligned}$$

Therefore, the inverse function is given by

$$f^{-1}(x) = x^2 + 2.$$

The domain of f is $\{x \mid x-2 \geq 0\}$. In interval notation, it is __________. Thus, the range of f^{-1} is also $[2, \infty)$.

Because $\sqrt{x-2}$ is nonnegative by the definition of square root. The range of f is $[0, \infty)$. Thus, the domain of f^{-1} is also __________.

Example 1.6.7. Consider the function $f(x) = 2(x+1)^3 - 1$. Find an inverse function f^{-1}.

Solution. To find the inverse function f^{-1}, we solve for y from the equation $f(y) = x$:

$$\begin{aligned} 2(y+1)^3 - 1 &= x \\ 2(y+1)^3 &= x+1 \\ (y+1)^3 &= \frac{x+1}{2} \\ y+1 &= \_\_\_\_\_\_\_\_\_\_ \\ y &= \sqrt[3]{\frac{x+1}{2}} - 1. \end{aligned}$$

Therefore, the inverse function is given by

$$f^{-1}(x) = \sqrt[3]{\frac{x+1}{2}} - 1.$$

The domain and range of f are both $(-\infty, \infty)$. Thus, the domain and range of f^{-1} are also both $(-\infty, \infty)$.

Exercises

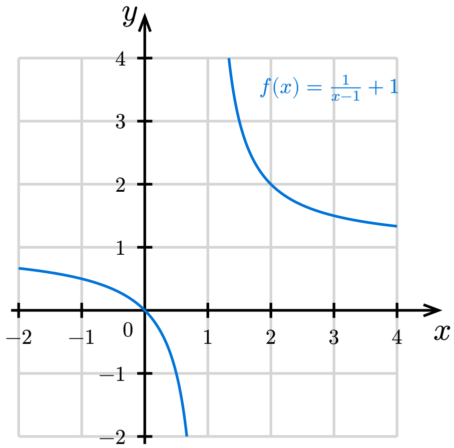
Exercise 1.6.1. Let f be a one-to-one function with $f(-2) = -3$ and $f(-3) = 4$. Find $f^{-1}(-3)$.

Answer: $f^{-1}(-3) = -2$.

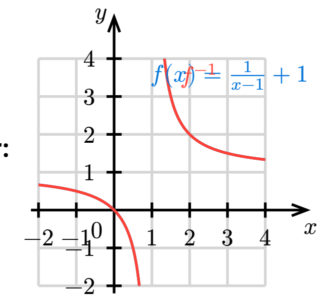
Exercise 1.6.2. Let $f(x) = x^3 - 1$ and $g(x) = \sqrt[3]{x+1}$. Is $g = f^{-1}$?

Answer: Yes, $g = f^{-1}$.

Exercise 1.6.3. Consider the function $f(x) = \frac{1}{x-1} + 1$ whose graph is shown below. Sketch the graph of f^{-1} without finding its equation.



Answer:



Exercise 1.6.4. Consider the function $f(x) = \frac{1-x}{x+1}$. Find the inverse function f^{-1} and its domain and range.

Answer: $f^{-1}(x) = \frac{1-x}{x+1}$; domain: $(-\infty,-1) \cap (-1,\infty)$; range: $(-\infty,-1) \cap (-1,\infty)$.

Exercise 1.6.5. Consider the function $f(x) = \sqrt[3]{\frac{x-1}{3}} + 2$. Find the inverse function f^{-1} and its domain and range.

Answer: $f^{-1}(x) = 3(x-2)^3 + 1$; domain: $(-\infty, \infty)$; range: $(-\infty, \infty)$.

Exercise 1.6.6. Consider the function $f(x) = \sqrt{x+1} - 1$. Find the inverse function f^{-1} and its domain and range.

Answer: $f^{-1}(x) = (x+1)^2 - 1$; domain: $[-1, \infty)$; range: $[-1, \infty)$.

Chapter 2 Polynomial and Rational Functions

2.1 Quadratic Functions and Applications

Definition 2.1.1 (Quadratic Functions)

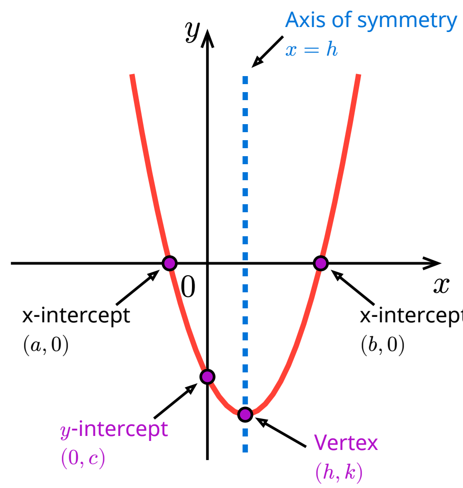
A function $f(x) = ax^2 + bx + c$ with $a \neq 0$ is called a **quadratic function**. Its graph is called a **parabola**.

By completing the square, a quadratic function can be written in the **standard form** (or **vertex form**):

$$f(x) = a(x-h)^2 + k, \quad \text{where } h = -\frac{b}{2a} \quad \text{and} \quad k = f(h).$$

The vertical line $x = -\frac{b}{2a}$ (or $x = h$) is called the **axis of symmetry**.

The **vertex** is the intersection of the axis of symmetry and the parabola and has the coordinates (h, k), equivalently, $\left(-\frac{b}{2a}, f\left(-\frac{b}{2a}\right)\right)$.



The ***y*-intercept** of a function f is the point $(0, f(0))$.

An ***x*-intercept** of a function f is the point $(x, 0)$, where x is a real solution of the equation $f(x) = 0$. If the equation $f(x) = 0$ has no real solution, then there is no x-intercept. ♣

Remark

Some textbooks refer an intercept as the non-zero coordinate rather than the point.

Example 2.1.1. Find the vertex form of the quadratic function $f(x) = 2x^2 + 4x + 1$ and determine the axis of symmetry, the vertex, x-intercepts, and the y-intercept of the function.

Solution. The axis of symmetry is

$$x = -\frac{\_\_\_\_\_}{2(\_\_\_\_\_)} = -1.$$

The vertex is $(-1, f(-1)) = (-1, \_\_\_\_\_)$.

The y-intercept is $(0, f(0)) = (0, \_\_\_\_\_)$.

To find the x-intercepts, we solve the equation $2x^2 + 4x + 1 = 0$. Using the quadratic formula, we have

$$x = \frac{-\_\_\_\_\_ \pm \sqrt{\_\_\_\_\_^2 - 4 \cdot \_\_\_\_\_ \cdot 1}}{2 \cdot 2} = \frac{-4 \pm \sqrt{8}}{4} = -1 \pm \frac{\sqrt{2}}{\_\_\_\_\_}.$$

Therefore, the x-intercepts are

$$\left(-1 + \tfrac{\sqrt{2}}{2}, 0\right) \quad \text{and} \quad \left(-1 - \tfrac{\sqrt{2}}{2}, 0\right).$$

Properties of Quadratic Functions

- The domain of a quadratic function is $(-\infty, \infty)$.
- If $a > 0$, then the parabola opens upward, the function has an absolute minimum $f(-\frac{b}{2a})$, and the range of the function is $[f(-\frac{b}{2a}), \infty)$.

  The function is increasing on the interval $(-\infty, -\frac{b}{2a})$ and decreasing on the interval $(-\frac{b}{2a}, \infty)$.
- If $a < 0$, then the parabola opens downward, the function has an absolute maximum $f(-\frac{b}{2a})$, and the range of the function is $(-\infty, f(-\frac{b}{2a})]$.

  The function is decreasing on the interval $(-\infty, -\frac{b}{2a})$ and increasing on the interval $(-\frac{b}{2a}, \infty)$.

Example 2.1.2. Find the range, the maximum, and the minimum of each function.

1) $f(x) = 3x^2 + 6x - 5$.

2) $f(x) = -2x^2 + 4 - 1$.

Solution. To find the range, and extremum of each function, we first find the y-coordinate of the vertex using the formula $f(-\frac{b}{2a})$.

1) For $f(x) = 3x^2 + 6x - 5$:

$$f\left(-\frac{b}{2a}\right) = f\left(-\frac{\underline{\qquad}}{2 \cdot 3}\right) = f(-1) = \underline{\qquad}.$$

Therefore, the range is $[\underline{\qquad}, \infty)$. Because $a = 3 > 0$, the minimum is $f(-1) = -8$. There is no maximum.

2) For $f(x) = -2x^2 + 4 - 1$:

$$f\left(-\frac{b}{2a}\right) = f\left(-\frac{\underline{\qquad}}{2 \cdot (-2)}\right) = f(0) = \underline{\qquad}.$$

Therefore, the range is $(-\infty, \underline{\qquad}]$. Because $a = -2 < 0$, the maximum is $f(0) = 3$. There is no minimum.

Example 2.1.3. A backyard farmer wants to enclose a rectangular space for a new garden within her fenced backyard. She has purchased 80 feet of wire fencing to enclose three sides, and she will use a section of the backyard fence as the fourth side. What's the maximal possible area of the garden.

Solution. Let x be the length of the side parallel to the backyard fence, and y be the length of the other two sides. Then the total length of the fencing used is given by the equation

$$x + 2y = 80,$$

which can be rewritten as

$$y = \underline{\qquad\qquad}.$$

The area A of the rectangular garden can be expressed as a function of x:

$$A(x) = x \cdot y = x \cdot \left(40 - \frac{x}{2}\right) = \underline{\qquad\qquad} .$$

The function $A(x)$ is a quadratic function with $a = -\frac{1}{2}$, $b = 40$, and $c = 0$. Since $a < 0$, the graph of $A(x)$ opens downward and has an absolute maximum at

$$x = -\frac{b}{2a} = -\frac{\underline{\qquad}}{2 \cdot \left(-\frac{1}{2}\right)} = \underline{\qquad} .$$

The maximal area is

$$A(40) = -\frac{1}{2} \cdot (\underline{\qquad})^2 + 40 \cdot (\underline{\qquad}) = 800\text{ft}^2.$$

Example 2.1.4. A ball is thrown upward from the top of a 40-foot-high building at a speed of 80 feet per second. The ball's height above ground can be modeled by the equation $H(t) = -16t^2 + 80t + 40$.

1) When does the ball reach the maximum height?
2) What is the maximum height of the ball?
3) When does the ball hit the ground? Round your answer to the nearest hundredth of a second.

Solution. The ball reaches the maximum height at time

$$t = -\frac{b}{2a} = -\frac{80}{2 \cdot (-16)} = \underline{\qquad} \text{ seconds.}$$

The maximum height of the ball is

$$H(\underline{\qquad}) = -16 \cdot (\underline{\qquad})^2 + 80 \cdot (\underline{\qquad}) + 40 = \underline{\qquad} \text{ feet.}$$

To find when the ball hits the ground, we solve the equation

$$-16t^2 + 80t + 40 = 0$$
$$2t^2 - 10t - 5 = 0.$$

Using the quadratic formula, we have

$$t = \frac{-\underline{\qquad} \pm \sqrt{\underline{\qquad}^2 - 4 \cdot 2 \cdot (\underline{\qquad})}}{2 \cdot 2} = \frac{-10 \pm \sqrt{60}}{4}.$$

Since time cannot be negative, we take the positive value:

$$t = \frac{-5 + \sqrt{\underline{\qquad}}}{2} \approx \underline{\qquad} \text{ seconds.}$$

2.2 Exercise

Exercise 2.2.1. For each of the following functions,

a) $f(x) = x^2 - 4x + 1.$

b) $f(x) = -2x^2 - 4x + 1.$

1) Write the function in vertex form,
2) Find the axis of symmetry,
3) Find the vertex,
4) Find the y-intercept,
5) Find the x-intercepts if they exist,
6) Find the domain and range,
7) Find the global maximum or minimum if it exists.

Answer:

a) Vertex form: $f(x) = (x-2)^2 - 3$; axis of symmetry: $x = 2$; vertex: $(2, -3)$; y-intercept: $(0, 1)$; x-intercepts: $\left(2 + \sqrt{3}, 0\right)$ and $\left(2 - \sqrt{3}, 0\right)$; domain: $(-\infty, \infty)$; range: $[-3, \infty)$; global minimum: -3 at $x = 2$.

b) Vertex form: $f(x) = -2(x+1)^2 + 3$; axis of symmetry: $x = -1$; vertex: $(-1, 3)$; y-intercept: $(0, -1)$; x-intercepts: $\left(-1 + \frac{\sqrt{6}}{2}, 0\right)$ and $\left(-1 - \frac{\sqrt{6}}{2}, 0\right)$; domain: $(-\infty, \infty)$; range: $(-\infty, 3]$; global maximum: 3 at $x = -1$.

Exercise 2.2.2. Find the dimensions of the rectangular parking lots producing the greatest area given that 500 feet of fencing will be used to for three sides.

Answer: 250 ft $\times$ 125 ft.

Exercise 2.2.3. A soccer stadium holds $62,000$ spectators. With a ticket price of $\$11$, the average attendance has been $26,000$. When the price dropped to $\$9$, the average attendance rose to $31,000$. Assuming that attendance is linearly related to ticket price, what ticket price would maximize revenue?

Answer: $\$10$.

Exercise 2.2.4. A toy rocket is launched in the air. Its height, in meters above sea level, as a function of time, in seconds, is given by $h(t) = -4.9t^2 + 2t + 5$.

1) Find the maximum height the rocket attains. Round your answer to the nearest hundredth meter.

2) When does the rocket reaches to 4 meters? Round your answer to the nearest hundredth second.

3) When does the rocket hit the ground? Round your answer to the nearest hundredth second.

Answer: 1) Approximately 5.20 meters. 2) Approximately 0.58 seconds and 1.75 seconds. 3) Approximately 2.28 seconds.

2.3 Polynomial Functions

Definition 2.3.1 (Power Functions)

A **power function** is a function that simplifies to the form $f(x) = ax^r$, where a is a non-zero constant (the **coefficient**), r is a real number (the **exponent**), and x is the independent variable

The domain of $f(x) = ax^r$ is usually all real numbers, but for some values of r it may be restricted to $x > 0$.

Example 2.3.1. Determine if the function is a power function.

1) $f(x) = -2x^3$ 2) $f(x) = \frac{1}{x^2}$ 3) $f(x) = \sqrt[3]{x}$ 4) $f(x) = 2^x$

Solution.

1) Yes, it is a power function with $a = -2$ and $r = 3$.

2) Yes, it is a power function with $a = 1$ and $r =$ ______.

3) Yes, it is a power function with $a = 1$ and $r =$ ______.

4) ______, it is an exponential function.

Definition 2.3.2 (End Behavior of Functions)

The **end behavior** of a function f describes what happens to $f(x)$ as x approaches positive or negative infinity.

If $f(x)$ approaches a fixed value b as x goes to ∞ or $-\infty$, then the horizontal line $y = b$ is called a **horizontal asymptote**.

Notation for End Behavior

We use an arrow $\to$ to mean "goes to" or "approaches."

For example, if $f(x) = x$, then the end behavior can be described as follows:

As $x \to \infty$, $f(x) \to \infty$. As $x \to -\infty$, $f(x) \to -\infty$.

How to Determine End Behavior

To determine the end behavior of f, choose some very large numbers $N > 0$ and evaluate $f(N)$ and $f(-N)$. The trend shows whether $f(x)$ approaches ∞, $-\infty$, or a finite value.

For example, if $f(1000)$, $f(10000)$ are all very large numbers, then as $x \to \infty$, $f(x) \to \infty$.

If the function is a power function, it is often convenient to plug in ∞ or $-\infty$ directly into the function to determine the end behavior.

Operations with Infinity

When working with ∞ and $-\infty$, use these rules:

$\infty + \infty = \infty$	$\infty + c = \infty$	$-\infty - \infty = -\infty$	$-\infty + c = -\infty$
$\infty \cdot c = \infty$ if $c > 0$	$\infty \cdot c = -\infty$ if $c < 0$	$-\infty \cdot c = -\infty$ if $c > 0$	$-\infty \cdot c = \infty$ if $c < 0$
$\infty \cdot \infty = \infty$	$-\infty \cdot -\infty = \infty$	$\infty^a = \infty$ $(a > 0)$	$\infty^a = 0$ $(a < 0)$

Note:

- The equal signs indicate trending behavior, not strict equality.
- $\infty + (-\infty)$ is indeterminate.
- $\frac{\infty}{\infty}$ is indeterminate.
- $(-\infty)^a$ only makes sense when a is a rational number with an odd denominator. In this case, $(-\infty)^a = -\infty$ if the numerator is odd, and $(-\infty)^a = \infty$ if the numerator is even.

Example 2.3.2. Determine the end behavior(s) of the function.

1) $f(x) = -2x^3$ 2) $f(x) = \frac{1}{x^2}$ 3) $f(x) = \sqrt[3]{x}$

Solution.

1) As $x \to \infty$, $f(x) \to -2 \cdot \infty^3 =$ ______.

As $x \to -\infty$, $f(x) \to -2 \cdot (-\infty)^3 =$ ______.

2) As $x \to \infty$, $f(x) \to \infty^{\_\_\_} = 0$.

As $x \to -\infty$, $f(x) \to (-\infty)^{\_\_\_} = 0$.

3) As $x \to \infty$, $f(x) \to \infty^{\frac{1}{3}} =$ ______.

As $x \to -\infty$, $f(x) \to (-\infty)^{\frac{1}{3}} =$ ______.

Definition 2.3.3 (Polynomial Functions)

Let n be an integer with $n \geq 0$. A **polynomial function** of **degree** n is a function that simplifies to the form:

$$f(x) = a_n x^n + \cdots + a_2 x^2 + a_1 x + a_0,$$

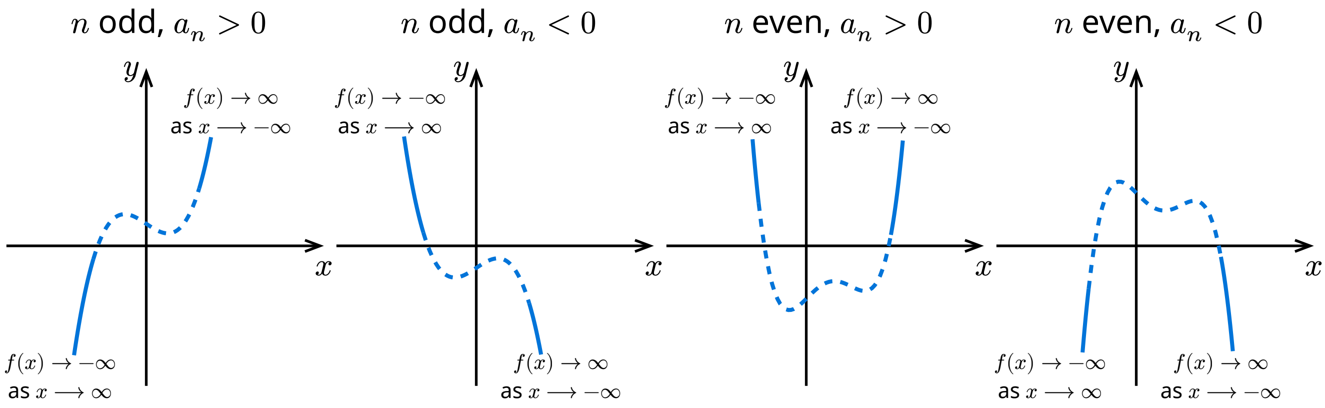
where a_i are real numbers for $i = 0, 1, \ldots, n$ and $a_n \neq 0$.

- Each a_i is a **coefficient**.
- Each product $a_i x^i$ is a **term** of the polynomial.
- The term $a_n x^n$ is the **leading term**, and a_n is the **leading coefficient**.
- The number a_0 is the **constant term**.

♣

Properties of Polynomial Functions

- The domain of a polynomial function is $(-\infty, \infty)$.
- The range of an odd degree polynomial function is also $(-\infty, \infty)$.
- The range of an even degree polynomial function is either $[y_{\min}, \infty)$ if $a_n > 0$ or $(-\infty, y_{\max}]$ if $a_n < 0$, where $y_{\min}$ and $y_{\max}$) are the absolute extrema.
- The end behavior of a polynomial function $f(x) = a_n x^n + \cdots + a_0$ of degree n is completely determined by the end behavior of the power function $g(x) = a_n x^n$.



Example 2.3.3. Determine the end behavior of the function using the arrow notation.

1) $f(x) = 2x^4 - 3x + 1$
2) $g(x) = x - 3x^3 + 2x^2$

Solution.

1) As $x \to \infty$, $f(x) \to 2 \cdot \infty^4 =$ ______.
As $x \to -\infty$, $f(x) \to 2 \cdot (-\infty)^4 =$ ______.

2) As $x \to \infty$, $g(x) \to -3 \cdot \infty^3 =$ ______.
As $x \to -\infty$, $g(x) \to -3 \cdot (-\infty)^3 =$ ______.

Example 2.3.4. Identify the degree, the leading therm and the end behavior of the polynomial function using the arrow notation.

1) $f(x) = -3x^2(x-1)(x+4)$
2) $f(x) = 2x^3(1-x)(x+1)$

Solution. To determine the degree and behavior, first simplify the expression.

1) $f(x) = -3x^2(x-1)(x+4) = -3x^2(x^2 +$ __________ $) - 3x^4 + 9x^3 + 12x^2$.

The degree is 4, the leading term is $-3x^4$.

As $x \to \infty$, $f(x) \to -3 \cdot \infty^4 =$ ______. As $x \to -\infty$, $f(x) \to -3 \cdot (-\infty)^4 =$ ______.

2) $f(x) = 2x^3(1-x)(x+1) = 2x^3($ __________ $+ 1) = -2x^5 + 2x^3 + 2x^2$.

The degree is 5, the leading term is $-2x^5$.

As $x \to \infty$, $f(x) \to -2 \cdot \infty^5 =$ ______. As $x \to -\infty$, $f(x) \to -2 \cdot (-\infty)^5 =$ ______.

Exercises

Exercise 2.3.1. Find the degree and leading coefficient, and determined the end behavior for the given polynomial.

1) $f(x) = -2x^4$ 2) $f(x) = 2x^5 - x^3$ 3) $f(x) = -2x(1 - x^2)$ 4) $f(x) = (x^2 - 1)(2x^4 - 1)$

Answer:

1) Degree: 4; leading coefficient: −2; end behavior: as $x \to \pm\infty$, $f(x) \to -\infty$.
2) Degree: 5; leading coefficient: 2; end behavior: as $x \to \pm\infty$, $f(x) \to \pm\infty$ respectively.
3) Degree: 3; leading coefficient: 2; end behavior: as $x \to \pm\infty$, $f(x) \to \pm\infty$ respectively.
4) Degree: 6; leading coefficient: 2; end behavior: as $x \to \pm\infty$, $f(x) \to \infty$.

2.4 Dividing of Polynomials

Theorem 2.4.1 (Euclidean Division Algorithm)

Let $p(x)$ and $d(x)$ be two polynomial. Suppose that $d(x)$ is non-zero and the degree of $d(x)$ is less than or equal to the degree of $f(x)$. Then there exist unique polynomials $q(x)$ and $r(x)$ such that

$$p(x) = d(x)q(x) + r(x)$$

and the degree of $r(x)$ is less than the degree of $d(x)$. ♡

Definition 2.4.2 (Long Division)

In the above theorem:

- $p(x)$ is the **dividend**,
- $d(x)$ is the **divisor**,
- $q(x)$ is the **quotient**,
- $r(x)$ is the **remainder**.

If $r(x) = 0$, then $d(x)$ **divides** $p(x)$. If $r(x) \neq 0$, then the degree of $r(x)$ is less than the degree of $d(x)$, and

$$\frac{p(x)}{d(x)} = q(x) + \frac{r(x)}{d(x)}.$$

A **division algorithm** computes the quotient and remainder.

The **long division** algorithm repeatedly applies Euclidean division to monomial quotients until the remainder has degree less than the divisor. ♣

Example 2.4.1. Divide $6x^3 + 11x^2 - 31x + 15$ by $3x - 2$.

Solution. We set up the long division as follows:

$$\begin{array}{rrrrrrrr}
 & & 2x^2 & + & \underline{\hspace{1.5em}} & - & 7 \\ \hline
\textcircled{3x} - 2 \,\big) & \boxed{6x^3} & + & 11x^2 & - & 31x & + & 15 \\
- & (6x^3 & - & 4x^2) & & & & \\ \hline
 & & & \boxed{\underline{\hspace{1.5em}}} & - & 31x & & \\
 & & - & (15x^2 & - & \underline{\hspace{1.5em}}) & & \\ \hline
 & & & & & \boxed{-21x} & + & \underline{\hspace{1.5em}} \\
 & & & & - & (\underline{\hspace{1.5em}} & + & 14) \\ \hline
 & & & & & & & 1
\end{array}$$

Explanation: In each step, we ***divide*** the leading term (***boxed***) of the current dividend (in odd-numbered rows) ***by*** the leading term (***circled***) of the divisor to ***get*** a term of the ***quotient*** (top row). Then we ***multiply*** the entire ***divisor by that quotient term*** and place it in the row below and ***subtract*** it from the current dividend to ***get*** the ***new dividend***. The ***last row*** is the ***remainder***.

Thus,

$$\frac{6x^3 + 11x^2 - 31x + 15}{3x - 2} = 2x^2 + 5x - 7 + \frac{1}{3x - 2}.$$

Example 2.4.2. Divide $4x^4 - x + 5$ by $x^2 - x + 3$.

Solution. We set up the long division as follows:

$$
\begin{array}{rrrrrrrr}
 & & & & 4x^2 & \underline{\qquad\qquad\qquad} & & \\
x^2 - x + 3 \,\big) & 4x^4 & + \; 0x^3 & + & 0x^2 & - & x & + \; 5 \\
- & (4x^3 & - \; 4x^3 & + & 12x^2) & & & \\
 & & 4x^3 & - & 12x^2 & - & x & \\
 & - & (4x^3 & \underline{\qquad\qquad\qquad}) & & & & \\
 & & & & \underline{\qquad\qquad} & & & + \; 5 \\
 & & & - & (-8x^2 + & 8x & + & \underline{\qquad}) \\
 & & & & & & \underline{\qquad} & + \; 29
\end{array}
$$

Thus,

$$\frac{4x^4 - x + 5}{x^2 - x + 3} = 4x^2 + 4x - 8 + \frac{\underline{\qquad\qquad}}{x^2 - x + 3}.$$

Definition 2.4.3 (Synthetic Division)

Synthetic division is a shortcut that can be used when the divisor is linear binomial in the form $x - c$. In synthetic division, only the coefficients are used in the division process. ♣

Example 2.4.3. Use synthetic division to divide $4x^3 + 10x^2 - 6x - 20$ by $x + 2$.

Solution. We set up the synthetic division as follows:

$$
\begin{array}{r|rrrr}
-2 & 4 & 10 & -6 & -20 \\
 & & -8 & \underline{\quad} & 20 \\
\hline
 & 4 & 2 & \underline{\quad} & \boxed{0}
\end{array}
$$

Explanation: We ***bring down the leading coefficient*** 4 to the third row. Then we ***multiply*** it by the ***zero of the divisor***, that is -2, and ***add*** it to the ***next coefficient*** 10 to get 2. We ***repeat*** this process until we reach the last coefficient. The ***last value*** (in boxed) is the ***remainder***.

Thus,

$$\frac{4x^3 + 10x^2 - 6x - 20}{x + 2} = 4x^2 + 2x - 10.$$

Example 2.4.4. Use synthetic division to divide $-9x^4 + 10x^3 + 7x^2 - 6$ by $x - 1$.

Solution. We set up the synthetic division as follows:

$$
\begin{array}{r|rrrrr}
1 & -9 & 10 & 7 & 0 & -6 \\
 & & -9 & \underline{\quad} & 8 & \underline{\quad} \\
\hline
 & -9 & 1 & \underline{\quad} & \underline{\quad} & \boxed{2}
\end{array}
$$

Thus,

$$\frac{-9x^4 + 10x^3 + 7x^2 - 6}{x - 1} = -9x^3 + x^2 + 8x + 8 + \frac{2}{x - 1}.$$

Theorem 2.4.4 (Reminder Theorem)

If a polynomial $f(x)$ is divided by $x - c$, then the remainder is the value $f(c)$. ♡

Proof. By the Euclidean Division Algorithm, we have

$$f(x) = (x - c)q(x) + r,$$

where r is the remainder. Since the degree of r is less than the degree of $x - c$, r must be a constant. Thus, we can write

$$f(x) = (x - c)q(x) + f(c).$$

Evaluating both sides at $x = c$, we have

$$f(c) = (c - c)q(c) + r = r.$$

□

Example 2.4.5. Use the Remainder Theorem to evaluate $f(x) = 6x^4 - x^3 - 15x^2 + 2x - 7$ at $x = 2$.

Solution. By the Remainder Theorem, the remainder when $f(x)$ is divided by $x - 2$ is $f(2)$. Using synthetic division, we have:

2	6	-1	-15	2	-7
		12	___	14	___
	6	11	___	___	___

Thus, $f(2) = 25$.

Exercises

Exercise 2.4.1. Divide $3x^2 - 7x - 3$ by $3x - 1$.

Answer: $\frac{3x^2-7x-3}{3x-1} = x - 2 - \frac{5}{3x-1}$.

Exercise 2.4.2. Divide $16x^3 - 12x^2 + 20x - 3$ by $4x + 5$.

Answer: $\frac{16x^3-12x^2+20x-3}{4x+5} = 4x^2 - 8x + 15 - \frac{78}{4x+5}$.

Exercise 2.4.3. Use synthetic division to divide $5x^3 - 3x - 36$ by $x - 3$.

Answer: $\frac{5x^2-3x-36}{x-3} = 5x^2 + 15x + 42 + \frac{90}{x-3}$.

Exercise 2.4.4. Divide $2x^4 + 4x^3 - 3x^2 - 5x - 2$ by $x + 2$.

Answer: $\frac{2x^4+4x^3-3x^2-5x-2}{x+2} = 2x^3 + 0x^2 - 3x + 1$.

Exercise 2.4.5. Use the Remainder Theorem to evaluate $f(x) = 2x^3 - 5x^2 + 4x - 1$ at $x = -1$.

Answer: $f(-1) = 12$.

2.5 Zeros of Polynomials

Definition 2.5.1 (Zeros of a Polynomial)

If f is a polynomial function, then a number c is called a **zero** of f if $f(c) = 0$. ♣

Theorem 2.5.2

Let f be a polynomial and c a real number. The following are equivalent:

1) c is a zero of f.
2) $x - c$ is a factor of $f(x)$.
3) $x = c$ is a solution of $f(x) = 0$.
4) $(c, 0)$ is an x-intercept of $y = f(x)$. ♡

Proof. The equivalence of (4) and (1) follows directly from the definition of x-intercept.

We show each remaining statement implies the next:

1) implies 2): Suppose c is a zero of f. Then $f(c) = 0$. By the Euclidean Division Algorithm, there exist polynomials $q(x)$ and $r(x)$ such that $f(x) = (x - c)q(x) + r$, where the degree of r is less than the degree of $x - c$. Since $f(c) = 0$, by the Remainder Theorem, $r = 0$. Therefore, $f(x) = (x - c)q(x)$, so $x - c$ is a factor of $f(x)$.

2) implies 3): Suppose $x - c$ is a factor of $f(x)$. Then there exists a polynomial $q(x)$ such that $f(x) = (x - c)q(x)$. By the zero-product property, if $f(x) = 0$, then $x - c = 0$, so $x = c$ is a solution.

3) implies 1): Suppose $x = c$ is a solution of $f(x) = 0$. Then $f(c) = 0$, so c is a zero of f. □

Example 2.5.1. Find x-intercepts and the y-intercept of the polynomial function

$$f(x) = x^3 + 3x^2 - x - 3.$$

Solution. To find the x-intercepts, we set $f(x) = 0$ and factor by grouping method:

$$\begin{aligned} f(x) = x^3 + 3x^2 - x - 3 &= (x^3 + 3x^2) + (-x - 3) \\ &= x^2(x + 3) + (\underline{\qquad})(x + 3) \\ &= (x + 3)(x^2 - 1) \\ &= (x + 3)(x + 1)(\underline{\qquad}). \end{aligned}$$

Solving $f(x) = 0$ by setting each factor equal to zero, we have: $x = -3$, $x = -1$, and $x = 1$. Thus, the x-intercepts are $(1, 0)$, $(-1, 0)$, and ________.

The y-intercept is found by evaluating $f(0)$:

$$f(0) = 0^3 + 3 \cdot 0^2 - 0 - 3 = -3.$$

Thus, the y-intercept is ________.

Definition 2.5.3 (Turning Point)

A **turning point** (also known as a local extremum) is a point at which the function values change from increasing to decreasing or decreasing to increasing. ♣

Theorem 2.5.4 (Fundamental Theorem of Algebra[1])

A degree n polynomial function has at least one complex zero. ♡

Corollary 2.5.4.1 (Maximal Number of Turning Points[2])

A degree n polynomial function may have at most n zeros and $n-1$ turning points. ♡

Example 2.5.2. Consider the polynomial function $f(x) = (x-2)(x+1)(x-4)$. Determine the zeros, the number of turning points, the x-intercepts, and the y-intercept.

Solution. The zeros of f are 2, -1, and ______.

Since the degree of f is 3, it may have at most ______ turning points.

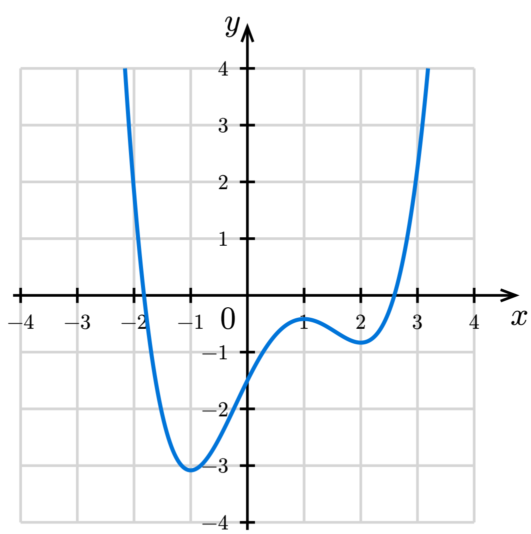
The x-intercepts are $(2,0)$, $(-1,0)$, and ________.

The y-intercept is found by evaluating $f(0)$:

$$f(0) = (0-2)(0+1)(0-4) = 8.$$

Thus, the y-intercept is $(0,8)$.

Example 2.5.3. What can we conclude about the leading term of the polynomial function $y = f(x)$ represented by the graph below.



Solution. Since as $x \to \pm\infty$, $f(x) \to \infty$, the leading coefficient is ______ and the degree must be ______.

Since the graph has three turning points, the degree of the polynomial is at least ______.

Thus, the leading term is of the form ax^n, where $a > 0$ and $n \geq 4$.

[1]A relatively elementary proof can be found at https://tinyurl.com/tFToA

[2]The corollary can be proved by induction together with facts of the derivative function.

Theorem 2.5.5 (Rational Zero Theorem)

Let $f(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0$ be polynomial with integer coefficients. Then every rational zero of $f(x)$ is in the form $\frac{p}{q}$, where p is a factor of the constant term a_0 and q is a factor of the leading coefficient a_n.

Example 2.5.4. List all possible rational zeros of $f(x) = 2x^4 - 5x^3 + x^2 - 4$.

Solution. The leading coefficient $a_4 = 2$ has factors ± 1 and ± 2.

The constant has factors ± 1, ± 2, and ______.

Thus, the possible rational zeros are

$$\pm\frac{1}{1} = \pm 1, \qquad \pm\frac{2}{1} = \pm\frac{4}{2} = \pm \underline{\qquad}, \qquad \pm\frac{4}{1} = \pm 4, \qquad \pm\frac{1}{2}.$$

Example 2.5.5. Find the zeros of $f(x) = 4x^3 - 3x - 1$.

Solution. Factoring the leading coefficient and the constant term, and then applying the Rational Zero Theorem, we have the possible rational zeros:

$$\pm 1, \qquad \pm\frac{1}{2}, \qquad \pm \underline{\qquad}.$$

Using synthetic division to test those possible zeros:

$$\begin{array}{r|rrrr} 1 & 4 & 0 & -3 & -1 \\ & & 4 & \underline{\quad} & \underline{\quad} \\ \hline & 4 & \underline{\quad} & \underline{\quad} & \boxed{0} \end{array}$$

We find that 1 is a zero of f. From the synthetic division, we have

$$f(x) = (x-1)(4x^2 + 4x + 1).$$

The rest of the zeros are found by solving $4x^2 + 4x + 1 = 0$. This quadratic equation can be solved using the rational zero theorem, factorization, completing the square, or the quadratic formula. Here, we use the factorization method:

$$\begin{aligned} 4x^2 + 4x + 1 &= 0 \\ (2x+1)(2x+1) &= 0 \\ 2x + 1 &= 0 \\ x &= \underline{\qquad}. \end{aligned}$$

Thus, the zeros of f are 1 and $-\frac{1}{2}$.

Theorem 2.5.6 (Linear Factorization)

Let $f(x)$ be a polynomial with the degree $n > 1$ and the leading coefficient a_n. Then
$$f(x) = a_n(x - c_1)(x - c_2)\cdots(x - c_n),$$
where c_i are complex numbers.

Proof. By the Fundamental Theorem of Algebra, $f(x)$ has at least one complex zero c_1. By the Factor Theorem, $x - c_1$ is a factor of $f(x)$. Thus, there exists a polynomial $q(x)$ such that
$$f(x) = (x - c_1)q(x).$$
The degree of $q(x)$ is $n - 1$. Repeating this process for $q(x)$, we can factor $f(x)$ completely into linear factors. □

Proposition 2.5.7 (Complex Conjugate Roots)

Let $f(x)$ be a polynomial with real coefficients. If $a + b\,i$ is a zero of f, then $a - b\,i$ is also a zero of f.

Proof. For a complex number $z = a + b\,i$, its conjugate is $\overline{z} = a - b\,i$. If $w = c + d\,i$ is another complex number, then
$$\overline{z + w} = \overline{z} + \overline{w}, \qquad \overline{z \cdot w} = \overline{z} \cdot \overline{w}.$$
Because the coefficients of f are real numbers, we have
$$\begin{aligned}\overline{f(z)} &= \overline{a_n \cdot z^n + a_{n-1} \cdot z^{n-1} + \cdots + a_1 \cdot z + a_0}\\ &= a_n \cdot \overline{z^n} + a_{n-1} \cdot \overline{z^{n-1}} + \cdots + a_1 \cdot \overline{z} + a_0\\ &= f(\overline{z}).\end{aligned}$$
Therefore, if $f(a + b\,i) = 0$, then $f(a - b\,i) = \overline{f(a + b\,i)} = 0$ which means that $a - b\,i$ is also a zero of f. □

Proposition 2.5.8 (Irrational Conjugate Roots[3])

Let $f(x)$ be a polynomial with rational coefficients. If $a + b\sqrt{m}$ is a zero, where a and b are rational numbers and $\sqrt{m}$ is irrational, then $a - b\sqrt{m}$ is also a zero.

Proof. The proof is similar to that of the Complex Conjugate Roots. We leave it as an exercise. □

Corollary 2.5.8.1 (Factorizations of Polynomials with Real Coefficients)

Let $f(x)$ be a polynomial with real coefficients. Then $f(x)$ can be factored into linear and irreducible quadratic factors with real coefficients.

Proof. This is a direct consequence of the Linear Factorization theorem and the Complex Conjugate Roots proposition. □

[3] Conjugates are fundamental in Galois theory. For more details, see the Wikipedia article Conjugate element (field theory).

Example 2.5.6. Find a fourth degree polynomial with real coefficients that has zeros of -3, 2, i, such that $f(-2) = 100$.

Solution. Since the polynomial has real coefficients and i is a zero, by the Complex Conjugate Roots proposition, $-i$ is also a zero. Because the degree of the polynomial is four, we have

$$\begin{aligned} f(x) &= a(x - \underline{\hspace{1.5cm}})(x - \underline{\hspace{1.5cm}})(x - \underline{\hspace{1.5cm}})(x - \underline{\hspace{1.5cm}}) \\ &= a(x+3)(x-2)(x^2+1). \end{aligned}$$

To determine the value of a, we use the condition $f(-2) = 100$:

$$\begin{aligned} f(-2) &= a(\underline{\hspace{1.5cm}} + 3)(-2-2)((-2)^2+1) \\ &= a(1)(-4)(\underline{\hspace{1.5cm}}) \\ &= -20a. \end{aligned}$$

From $f(-2) = 100$, we have

$$\begin{aligned} -20a &= 100 \\ a &= \underline{\hspace{1.5cm}}. \end{aligned}$$

Thus, an equation for f is

$$f(x) = -5(x+3)(x-2)(x^2+1).$$

Example 2.5.7. Let $f(x) = x^4 + 2x^2 - 8$.

1) Factor f into linear and irreducible quadratic factors with real coefficients.

2) Factor f completely into linear factors with complex coefficients.

Solution. Note that $x^4 = \left(x^2\right)^2$, so we can treat f as a quadratic in x^2.

First, factor f using undetermined coefficients:

$$\begin{aligned} f(x) &= x^4 + 2x^2 - 8 \\ &= (x^2 - \underline{\hspace{1.5cm}})(x^2+4). \end{aligned}$$

Solving $x^2 - 2 = 0$ gives two linear factors with real coefficients:

$$x^2 - 2 = \left(x - \sqrt{2}\right)(x + \underline{\hspace{1.5cm}}).$$

Solving $x^2 + 4 = 0$ gives two linear factors with complex coefficients:

$$x^2 + 4 = (x - 2\,i)(x + \underline{\hspace{1.5cm}}).$$

1) To factor f into linear and irreducible quadratic factors with real coefficients, keep x^2+4 as is and factor x^2-2:

$$f(x) = \left(x - \sqrt{2}\right)\left(x + \sqrt{2}\right)(x^2+4).$$

1) To factor f completely into linear factors with complex coefficients, use all four linear factors:

$$f(x) = \left(x - \sqrt{2}\right)\left(x + \sqrt{2}\right)(x - 2\,i)(x + 2\,i).$$

Exercises

Exercise 2.5.1. Find x-intercepts (if they exist) and the y-intercept of the polynomial function.

1) $f(x) = -2x^4 + x^2 + 1$

2) $f(x) = x^3 + x^2 - 4x - 4$

Answer: **1)** x-intercepts: none; y-intercept: $(0, 1)$. **2)** x-intercepts: $(-2, 0)$ and $(2, 0)$; y-intercept: $(0, -4)$.

Exercise 2.5.2. Find all zeros of $f(x) = 2x^3 + 5x^2 - 11x + 4$.

Answer: The zeros are $\frac{1}{2}$, -4, and 1.

Exercise 2.5.3. Find a fourth degree polynomial with real coefficients that has zeros of -1, 2, $1+i$, such that $f(-2)=10$.

Answer: $f(x)=-(x+1)(x-2)(x^2-2x+2)$.

Exercise 2.5.4. Let $f(x)=x^3-5x^2+6x-30$.

1) Factor f into linear and irreducible quadratic factors with real coefficients.

2) Factor f completely into linear factors with complex coefficients.

Answer: **1)** $f(x)=(x-3)(x^2-2x+10)$. **2)** $f(x)=(x-3)(x-1-3\,i)(x-1+3\,i)$.

2.6 Graphs of Polynomials

Definition 2.6.1 (Multiplicity of a Zero)

We say a zero c of a polynomial function f has the **multiplicity** k if $f(x) = (x-c)^k g(x)$ and c is not a zero of g.

Example 2.6.1. Find the zeros of the polynomial function $f(x) = x^3(x-1)^2(x-2)$ and determine their multiplicities.

Solution. By the Factor Theorem, we see that the zeros of f are

$$0, \quad \underline{\qquad}, \quad \text{and} \quad 2.$$

By the definition multiplicity, we have

Zeros:	0	1	2
Multiplicities:	3	____	____

Example 2.6.2. A polynomial function f of degree 3 has two zeros 1 and 2 with multiplicity 2 and 1 respectively. The y-intercept is $(0,-4)$. Find an equation for P.

Solution. Since the zero 1 has multiplicity 2 and the zero 2 has multiplicity 1, we have

$$f(x) = (x-1)^2(x-2)q(x).$$

Because the degree of f is 3, the degree of $q(x)$ must be 0, that is, $q(x) = a$, where a is a constant. Thus, we have

$$f(x) = a(x - \underline{\qquad})^2(x - \underline{\qquad}).$$

To determine the value of a, we use the condition that the y-intercept is $(0,-4$:

$$f(0) = a(0-1)^2(0-2) = \underline{\qquad}.$$

From $f(0) = \underline{\qquad}$, we have $a = 2$.

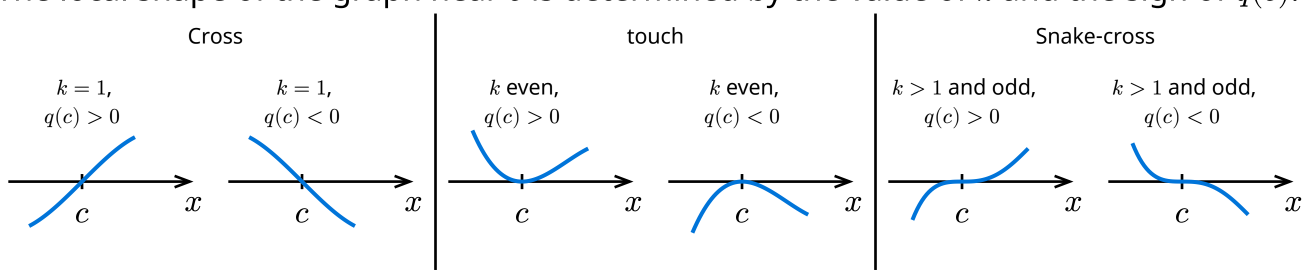
Thus, an equation for f is

$$f(x) = \underline{\qquad\qquad\qquad}.$$

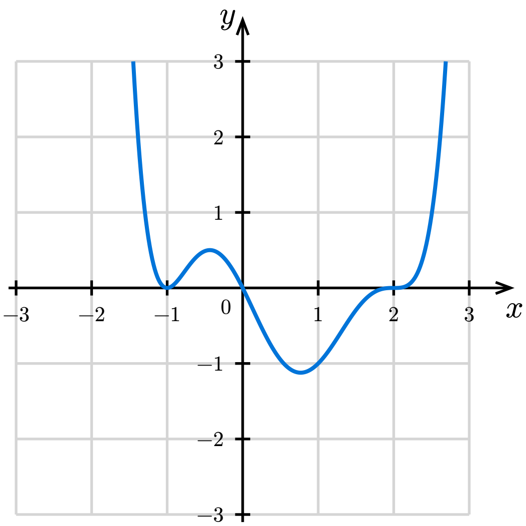
Local Graph Near a Zero

Let f be a polynomial with positive leading coefficient and c is a zero of f of multiplicity k. Write $f(x) = (x-c)^k q(x)$

The local shape of the graph near c is determined by the value of k and the sign of $q(c)$:



Example 2.6.3. Use the graph of the function of degree 6 in the figure below to identify the zeros of the function and their possible multiplicities.



Solution. The zeros of the function are -1, 0, and 2.

Because the graph touches the x-axis at $x = -1$, the multiplicity is is even.

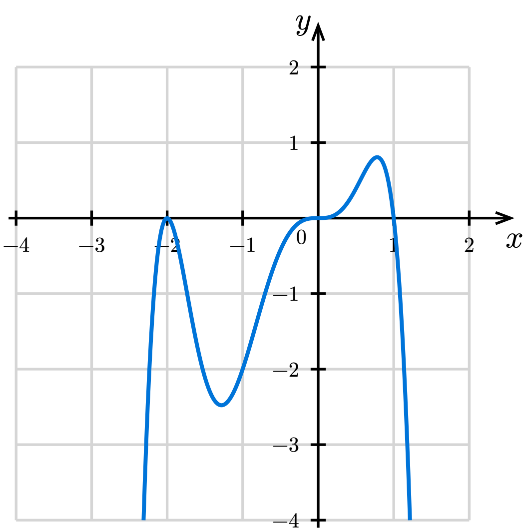
Because the graph crosses the x-axis t $x = 0$, the multiplicity is ______.

Because the graph snake-crosses the x-axis at $x = 2$, the multiplicity is odd and greater than 1.

Because the degree of the polynomial is 6, the zeros and their multiplicities are:

Zeros:	-1	0	2
Multiplicities:	______	1	______

Example 2.6.4. Find a polynomial of the least degree whose graph is given below.



Solution. From the graph, we see that the zeros are -2, 0, and 1.

Because the graph touches the x-axis at $x = -2$, the multiplicity is at least ______.

Because the graph crosses the x-axis t $x = 0$, the multiplicity is 1.

Because the graph snake-crosses the x-axis at $x = 1$, the multiplicity is at least ______.

Thus, an equation for the polynomial of least degree is

$$f(x) = ax^3(x+2)^2(x-1).$$

Since the point $(-1, -2)$ is on the graph, the coefficient a satisfies the equation

$$f(-1) = a(-1+2)^2(-1)^2(-1-1) = -2.$$

Solving for a gives $a =$ ______. Therefore, an equation for the polynomial is

$$f(x) = -x^3(x+2)^2(x-1).$$

Definition 2.6.2 (Continuity of Polynomials)

A function is **continuous** on an interval if its graph has no breaks there. It is **smooth** on an interval if its graph has no breaks and no sharp corners.

A function is continuous (respectively, smooth) if it is continuous (respectively, smooth) on every interval in its domain.

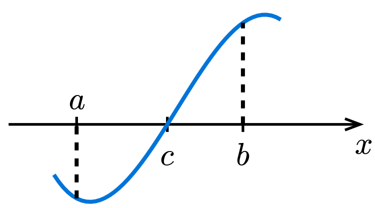
Proposition 2.6.3

Polynomial functions and rational functions are smooth functions.

Theorem 2.6.4 (Intermediate Value Theorem)

If f is continuous on $[a, b]$ and $f(a)f(b) < 0$, then there exists at least one c between a and b such that $f(c) = 0$.

In particular, this holds for polynomial and rational functions.



Corollary 2.6.4.1

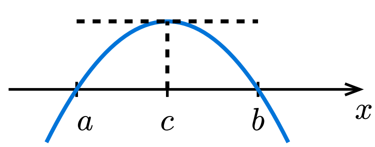
Let f be a polynomial function, and let a and b be real zeros of f. If f has no other zeros between a and b, then either $f(x) > 0$ for all x between a and b, or $f(x) < 0$ for all x between a and b.

Proof. Assume, for contradiction, that there exist c_1 and c_2 between a and b such that $f(c_1)f(c_2) < 0$. By the Intermediate Value Theorem, there is at least one d between c_1 and c_2 with $f(d) = 0$. This contradicts the assumption that f has no other zeros between a and b. Therefore, the corollary holds. □

Theorem 2.6.5 (Rolles's Theorem)

Let f be a smooth function, a and b two zeros. Then f has at lease one local extremum (turning point) between a and b.

In particular, this holds for polynomial and rational functions.



Remark

Continuity and smoothness are key concepts in Calculus. The Intermediate Value Theorem and Rolle's Theorem are fundamental tools, but their proofs require concepts of **limit** and **derivative**, which are beyond the scope of this course.

Example 2.6.5. Determine if the polynomial function $f(x) = 5x^4 - 2x^3 - 20$ has a zero on the interval $[1, 2]$.

Solution. Since $1^n = 1$, it's easier to compute $f(1)$:

$$f(1) = 5(1)^4 - 2(1)^3 - 20 = \underline{\qquad}.$$

For $f(2)$, we can use the Remainder Theorem together with synthetic division:

2	5	−2	0	0	−20
		10	____	____	____
	5	____	16	____	**44**

Since $f(1)f(2) < 0$ and f is continuous over $[1, 2]$, by the Intermediate Value Theorem, there exists at least one value c between 1 and 2 such that $f(c) = 0$.

Example 2.6.6. The function $f(x) = x^3 - 4x$ has the zeros $x = -2$, $x = 0$, and $x = 2$. Determine the intervals over which $f(x) > 0$.

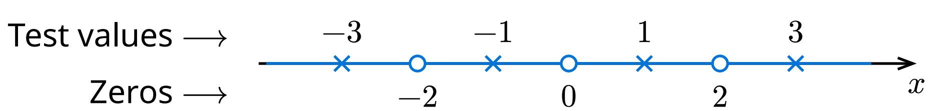
Solution.

Because the degree of f is ______, by the Fundamental Theorem of Algebra, f has at most 3 real zeros. Therefore, there are no zeros other than the given ones.

The zeros divide the real line into four intervals: $(-\infty, -2)$, $(-2, 0)$, $(0, 2)$, and $(2, \infty)$.

Since f is continuous, by the corollary of the Intermediate Value Theorem, the sign of $f(x)$ does not change within each interval.

To determine where $f(x) > 0$, test a point in each interval. The test values and zeros are shown in the figure below. The open circles indicate that the zeros are not included because the inequality sign is $>$.



- For $(-\infty, -2)$, let $x = -3$, then $f(-3)$ ________ 0.
- For $(-2, 0)$, let $x = -1$, then $f(-1)$ ________ 0.
- For $(0, 2)$, let $x = 1$, then $f(1)$ ________ 0.
- For $(2, \infty)$, let $x = 3$, then $f(3)$ ________ 0.

Therefore, $f(x) > 0$ on the intervals ________ $\cup$ ________.

Definition 2.6.6 (Guidelines for Graphing Polynomial Functions)

1) Plot the y-intercept.
2) Find real zeros and their multiplicities; sketch the local graph near each x-intercept.
3) Determine end behavior and sketch the left and right tails.
4) Test values to check whether the graph lies above or below the x-axis between zeros; estimate turning points.
5) Connect points and local graphs smoothly.

♣

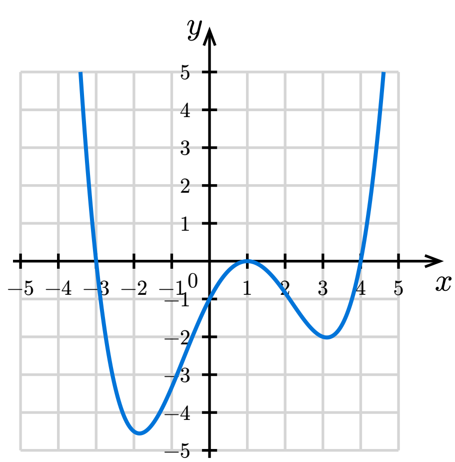
Example 2.6.7. Sketch the graph of the polynomial function

$$f(x) = \frac{1}{12}(x-4)(x-1)^2(x+3).$$

Solution.

1) The y-intercept is $f(0) = \frac{1}{12}(0-4)(0-1)^2(0+3) =$ ______.
2) The real zeros are -3, 1, and 4 with multiplicities 1, 2, and 1 respectively. The local graphs near the x-intercepts are shown below.
3) Since the degree of f is 4 (even) and the leading coefficient is positive, the end behavior is: as $x \to -\infty$, $f(x) \to \infty$; and as $x \to \infty$, $f(x) \to \infty$.
4) To determine whether the graph lies above or below the x-axis between zeros, test values in each interval:
   - For $(-\infty, -3)$, let $x = -4$, then $f(-4) = \frac{1}{12}(-4-4)(-4-1)^2(-4+3) =$ ______.
   - For $(-3, 1)$, let $x = 0$, then $f(0) = \frac{1}{12}(0-4)(0-1)^2(0+3) =$ ______.
   - For $(1, 4)$, let $x = 2$, then $f(2) = \frac{1}{12}(2-4)(2-1)^2(2+3) =$ ______.

   Therefore, the graph lies below the x-axis on __________, and above on __________.
5) Using this information, we can sketch the graph of f as shown below.



Exercises

Exercise 2.6.1. Find the zeros and their multiplicities of the polynomial function

$$f(x) = 3x^4 - 15x^3 + 12x^2.$$

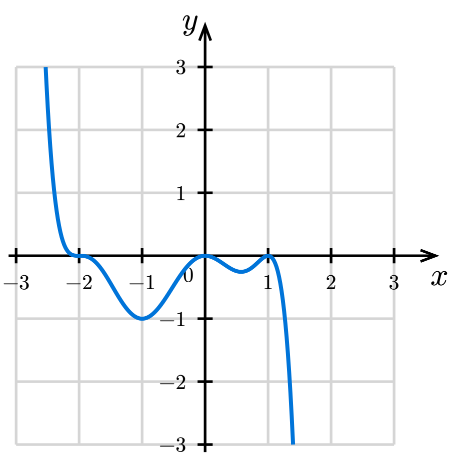
Answer:

Zeros:	0	1	4
Multiplicities:	2	1	1

Exercise 2.6.2. A polynomial function P of degree 4 has two zeros 1 and 2 with multiplicity 3 and 1 respectively. The y-intercept is $(0, -4)$. Find an equation for P.

Answer: $P(x) = -2(x-1)^3(x-2)$.

Exercise 2.6.3. Find a polynomial of the least degree whose graph is given below.

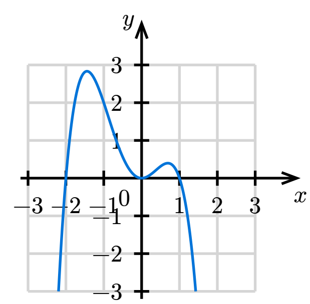


Answer: $f(x) = -\frac{1}{4}x^2(x-1)^2(x+2)^3$.

Exercise 2.6.4. Sketch the graph of the polynomial function

$$f(x) = -x^4 - x^3 + 2x^2.$$

Answer:



2.7 Rational Functions

Definition 2.7.1 (Rational Functions)

Let $p(x)$ and $q(x)$ be polynomials with $\deg(q(x)) > 0$. The function $f(x) = \frac{p(x)}{q(x)}$ is called a rational function. The domain of f is $\{x \mid q(x) \neq 0\}$. ♣

Example 2.7.1. Find the domain of $f(x) = \frac{x+3}{x^2-9}$ in interval notation.

Solution. To find the domain of f, we need to find the values of x such that the denominator is not zero. We have

$$\begin{aligned} x^2 - 9 &= 0 \\ (x-3)(x+3) &= 0 \\ x = \underline{\qquad} \quad &\text{or} \quad x = \underline{\qquad}\ . \end{aligned}$$

Therefore, the domain of f is $(-\infty, \underline{\qquad}) \cup (-3, \underline{\qquad}) \cup (3, \underline{\qquad})$.

Definition 2.7.2 (Asymptotes and Holes)

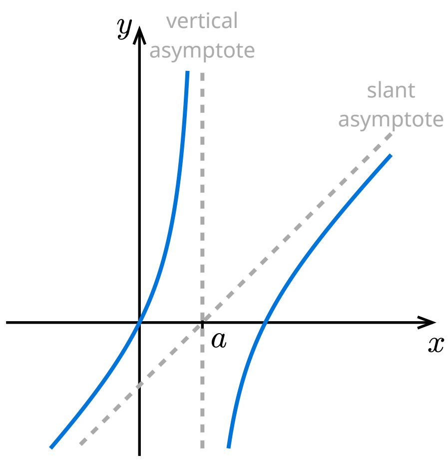
A **vertical asymptote** of a function f is a vertical line $x = a$ where the graph of f approaches positive or negative infinity as x approaches a from the left or right. In other words, as $x \to a^-$ or a^+, $f(x) \to \infty$ or $f(x) \to -\infty$, where $x \to a^-$ (or a^+) means x approaches a from the left (respectively, right).

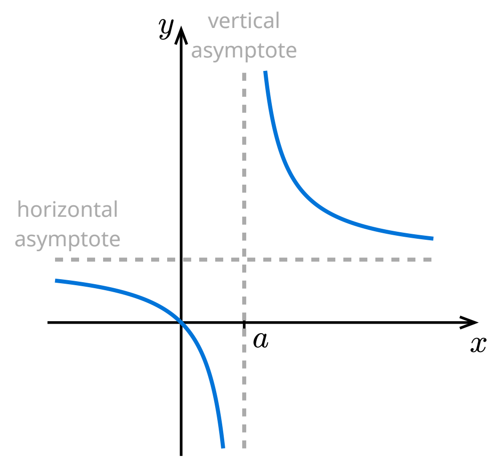
A function f has a **removable discontinuity** (or **hole**) at $x = a$ if $f(x) \to b$ as $x \to a$ but $f(a)$ is undefined.

Let $f = \frac{p(x)}{q(x)}$ be a rational function:

- If $p(a) = q(a) = 0$, then f has a hole at a.
- If $q(a) = 0$ but $p(a) \neq 0$, then f has a vertical asymptote at $x = a$.

A **slant (oblique) asymptote** of a function f is a line $y = mx + b$ with $m \neq 0$ where the graph of f approaches $mx + b$ as x goes to positive or negative infinity. That is, as $x \to \infty$ or $x \to -\infty$, $f(x) \to mx + b$.





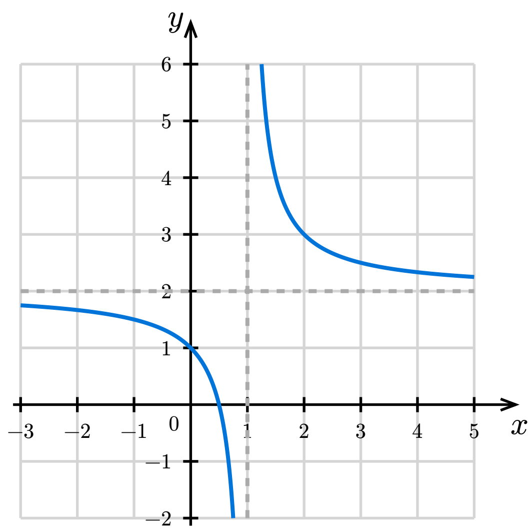
♣

How to Find Horizontal and Slanted Asymptotes

Let $f(x) = \dfrac{p(x)}{q(x)} = \dfrac{a_m x^m + a_{m-1}x^{m-1} + \ldots + a_1 x + a_0}{b_n x^n + b_{n-1}x^{n-1} + \ldots + b_1 x + b_0}$ be a rational function.

- If $m < n$, then f has a horizontal asymptote $x = 0$;
- If $m = n$, then f has a horizontal asymptote $x = \frac{a_m}{b_n}$;
- If $m = n + 1$, then f has a slated asysmptote $y = mx + b$, where $mx + b$ is the quotient of $p(x)$ divided by $q(x)$.
- If $m > n + 1$, then f has no horizontal or slated asymptote;

Example 2.7.2. Find equations for the asymptotes of the function f graphed in the figure below.

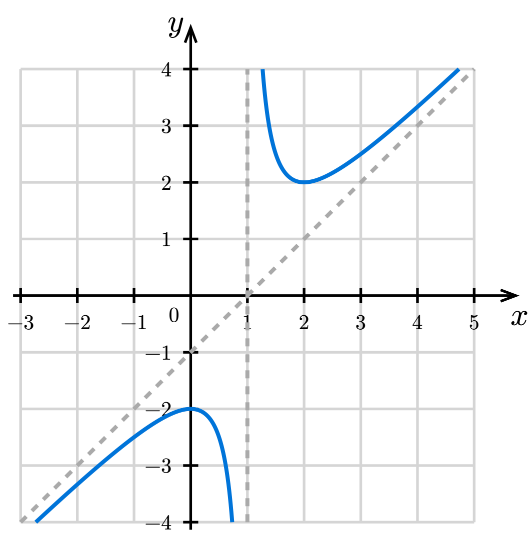


Solution. From the graph, we observe:

- As $x \to -\infty$, $f(x) \to 2$.
- As $x \to \infty$, $f(x) \to$ ______.
- As $x \to 1^-$, $f(x) \to -\infty$.
- As $x \to 1^+$, $f(x) \to$ ______.

Therefore, f has a horizontal asymptote ________ and a vertical asymptote ________.

Example 2.7.3. Find equations for the asymptotes of the function f graphed in the figure below.



Solution. As $x \to 1^{\pm}$, the graph approaches the vertical line $x =$ ______. Therefore, f has a vertical asymptote ________.

As $x \to \pm\infty$, the graph approaches the slanted line. Therefore, f has a slanted asymptote.

Because the slanted line passes through

$$(\_\_\_\_\_\_, 0) \quad \text{and} \quad (0, \_\_\_\_\_\_),$$

an equation of the slanted line is

$$y = \_\_\_\_\_\_\ x - 1.$$

Example 2.7.4. Find the asymptotes of the function $f(x) = \dfrac{x^2+1}{2x^2-3x+1}$.

Solution. Since the degree of the numerator is equal to the degree of the denominator, f has a horizontal asymptote at

$$y = \underline{\hspace{2cm}}.$$

To find vertical asymptotes, set the denominator equal to zero and solve for x:

$$\begin{aligned} 2x^2-3x+1 &= 0 \\ (2x-1)(x-1) &= 0 \\ x = \underline{\hspace{2cm}} \quad \text{or} \quad x &= \underline{\hspace{2cm}}. \end{aligned}$$

Therefore, f has vertical asymptotes at

$$x = \underline{\hspace{2cm}} \quad \text{and } x = \underline{\hspace{2cm}}.$$

Example 2.7.5. Find the asymptotes of the function $f(x) = \dfrac{-x^2+3x-1}{x-1}$.

Solution. Since the degree of the numerator is one more than the degree of the denominator, f has a slant asymptote. Using polynomial long division or synthetic division, we have

$$\frac{-x^2+3x-1}{x-1} = \underline{\hspace{4cm}} + \frac{1}{x-1}.$$

Therefore, an equation of the slant asymptote is

$$y = \underline{\hspace{2cm}}\, x + \underline{\hspace{2cm}}.$$

To find vertical asymptotes, set the denominator equal to zero and solve for x:

$$\begin{aligned} x-1 &= 0 \\ x &= \underline{\hspace{2cm}}. \end{aligned}$$

Therefore, f has a vertical asymptote at

$$x = \underline{\hspace{2cm}}.$$

Example 2.7.6. Find the asymptotes and holes of the function $f(x) = \dfrac{x^2+x-6}{x^3-2x^2-x+2}$.

Solution. Factor the numerator and denominator:

$$f(x) = \frac{(x+3)(x-2)}{(x-2)(x+1)(x-1)}.$$

Since $x-2$ is a common factor in the numerator and denominator, f has a hole at

$$x = \underline{\hspace{2cm}}.$$

The vertical asymptotes are determined by zeros of the denominator after canceling common factors. The denominator of the reduced form of $f(x)$ is $(x+1)(\underline{\hspace{2.5cm}})$. Therefore, the vertical asysmptotes of f are

$$x = \underline{\hspace{2cm}} \quad \text{and} \quad x = \underline{\hspace{2cm}}.$$

Since the degree of the numerator is less than the degree of the denominator, f has a horizontal asymptote at

$$y = \underline{\hspace{2cm}}.$$

Definition 2.7.3 (Guidelines for Graphing Rational Functions)

1) Find the y-intercept and plot it.
2) Find the x-intercept(s) and plot them.
3) Identify all vertical asymptotes and draw them as dashed lines.
4) Determine whether the function has a horizontal or slant asymptote (or neither), and draw the asymptote as a dashed line.
5) In each interval between consecutive zeros of the denominator or the function, choose a test point to determine whether the graph lies above or below the x-axis.
6) Sketch the graph using all the information above.

♣

Example 2.7.7. Sketch a graph of $f(x) = \dfrac{(x+2)(x-3)}{(x+1)^2(x-2)}$.

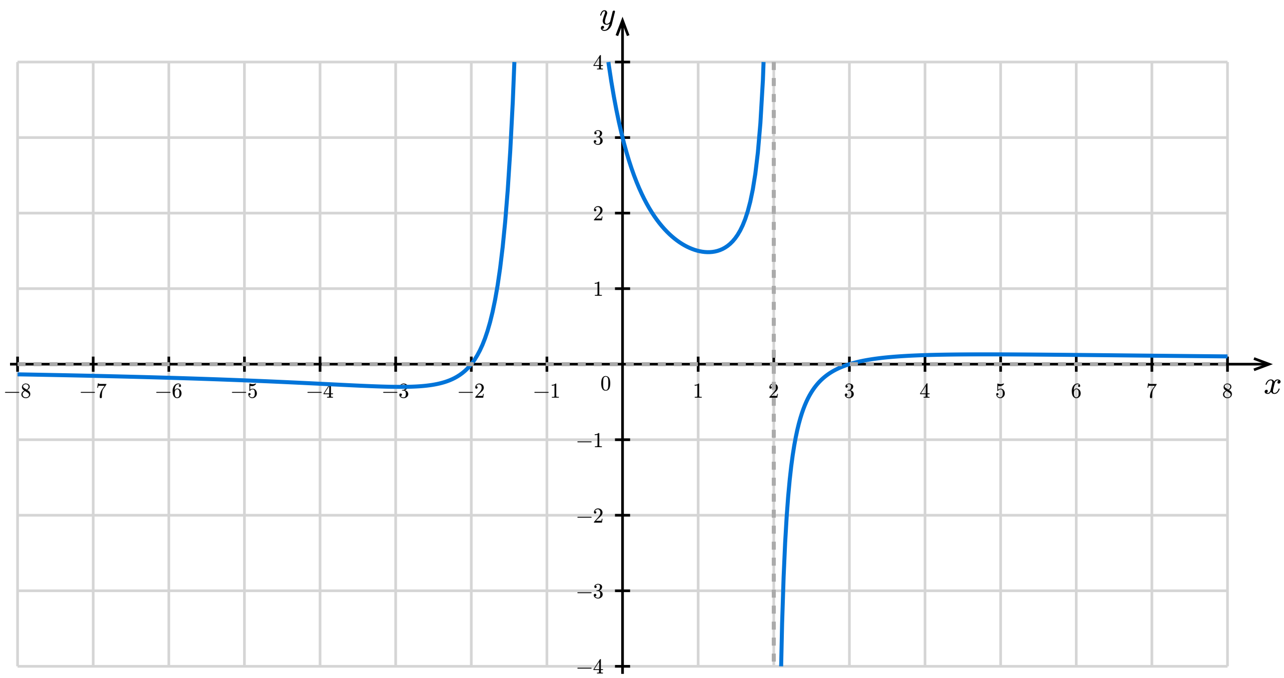
Solution.

1) The y-intercept is $(0, f(0)) =$ ________.
2) The x-intercepts are found by setting the numerator equal to zero:
$$(x+2)(x-3) = 0$$
$$x = \underline{\qquad} \quad \text{or} \quad x = \underline{\qquad}.$$
Therefore, the x-intercepts are ________ and ________.
3) The vertical asymptotes are found by setting the denominator equal to zero:
$$(x+1)^2(x-2) = 0$$
$$x = \underline{\qquad} \quad \text{or} \quad x = \underline{\qquad}.$$
Therefore, the vertical asymptotes are ________ and ________.
4) Since the degree of the numerator is less than the degree of the denominator, f has a horizontal asymptote at
$$y = \underline{\qquad}.$$
5) To determine whether the graph lies above or below the x-axis in each interval determined by the vertical asymptotes and x-intercepts, we choose test values:
   - For $(-\infty, -2)$, let $x = -3$, then
   $$f(-3) = \frac{(-3+2)(-3-3)}{(-3+1)^2(-3-2)} \underline{\qquad} 0.$$
   - For $(-2, -1)$, let $x = -1.5$, then
   $$f(-1.5) = \frac{(-1.5+2)(-1.5-3)}{(-1.5+1)^2(-1.5-2)} \underline{\qquad} 0.$$
   - For $(-1, 2)$, let $x = 0$, then
   $$f(0) = \frac{(0+2)(0-3)}{(0+1)^2(0-2)} \underline{\qquad} 0.$$
   - For $(2, \infty)$, let $x = 3$, then

$$f(3) = \frac{(3+2)(3-3)}{(3+1)^2(3-2)} \_\_\_\_\_\_ 0.$$

Therefore, the graph lies above the x-axis on __________ ∪ __________, and below on __________ ∪ __________.

6) Using this information, we can sketch the graph of f as shown below.



Exercises

Exercise 2.7.1. Find asymptotes of the rational function

$$f(x) = \frac{3x^2 - 1}{x^2 + 4x - 5}.$$

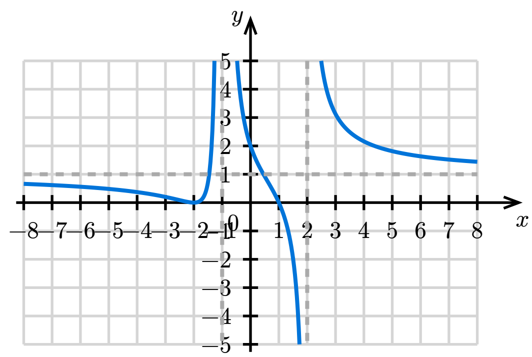
Answer: Horizontal asymptote: $y = 3$. Vertical asymptotes: $x = 1$ and $x = -5$.

Exercise 2.7.2. Find asymptotes of the rational function $f(x) = \frac{x^2}{x+1}$.

Answer: Slant asymptote: $y = x - 1$. Vertical asymptote: $x = -1$.

Exercise 2.7.3. Sketch a graph of the rational function $f(x) = \frac{(x+2)^2(x-1)}{(x+1)^2(x-2)}$.

Answer:



2.8 Nonlinear Inequalities

Definition 2.8.1 (Guidelines for Solving Polynomial or Rational Inequalities)

1) Rewrite the inequality in one of the following forms, according to the original inequality: $f(x) > 0$, $f(x) \geq 0$, $f(x) < 0$, or $f(x) \leq 0$.
2) Find all real zeros of $f(x)$.
3) Divide the **DOMAIN OF THE FUNCTION** (the entire real line if f is a polynomial) into intervals using the zeros found in the previous step.
4) Choose a test point from each interval to determine the sign of $f(x)$.
5) Identify the solution set as the union of intervals where the test point satisfies the inequality, and decide whether to include the boundary points. ♣

Example 2.8.1. Solve the inequality $x^2 \leq 7x - 6$.

Solution. Rewrite the inequality as

$$\underline{\hspace{3cm}} \leq 0.$$

To find the zeros of $f(x) = x^2 - 7x + 6$, set $f(x) = 0$:

$$x^2 - 7x + 6 = 0$$
$$(x-6)(x-1) = 0$$
$$x = \underline{\hspace{1.5cm}} \quad \text{or} \quad x = \underline{\hspace{1.5cm}}.$$

The zeros divide the real line into three intervals:

$$(-\infty, 1), \quad (1, 6), \quad \text{and} \quad (6, \infty).$$

Choose a test point from each interval to ***determine the sign*** of $f(x)$:

- For $(-\infty, 1)$, let $x = 0$, then $f(0) =$ ____________________.
- For $(1, 6)$, let $x = 3$, then $f(3) =$ ____________________.
- For $(6, \infty)$, let $x = 7$, then $f(7) =$ ____________________.

Therefore, the solution set is _______________ ∪ _______________.

Example 2.8.2. Solve the inequality $\dfrac{6x}{(x+1)(x+2)} \geq 1$.

Solution. Rewrite the inequality as

$$\frac{6x}{(x+1)(x+2)} - 1 \geq 0.$$

Let

$$f(x) = \frac{6x}{(x+1)(x+2)} - 1 = \frac{\underline{\hspace{3cm}}}{(x+1)(x+2)}.$$

We need to find the zeros and the domain of f.

To find the zeros of $f(x)$, set the numerator equal to zero and solve for x:

$$\begin{aligned} -x^2+3x-2&=0 \\ (x-1)(x-2)&=0 \\ x = \underline{\hspace{2cm}} \quad \text{or} \quad x &= \underline{\hspace{2cm}}. \end{aligned}$$

To find the domain of $f(x)$, set the denominator equal to zero and solve for x:

$$\begin{aligned} (x+1)(x+2)&=0 \\ x = \underline{\hspace{2cm}} \quad \text{or} \quad x &= \underline{\hspace{2cm}}. \end{aligned}$$

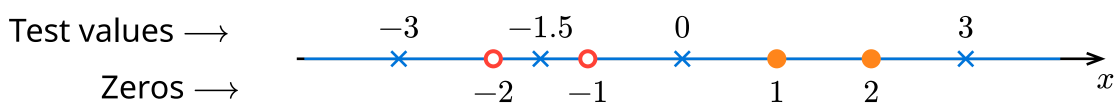
The domain of f is

$$(-\infty,-2)\cup(-2,-1)\cup(-1,\infty).$$

The zeros divide the domain of f into four intervals:

$$(-\infty,-2), \quad (-2,-1), \quad (-1,1], \quad [1,2] \quad \text{and} \quad [2,\infty).$$

Note the reason we use square brackets for the last two intervals is that the original inequality is a "greater than or equal to" inequality. Correspondingly, circles are used for an open boundary point and solid dots for a closed boundary point in the figure below.



Choose a test point from each interval to ***determine the sign*** of $f(x)$:

- For $(-\infty,-2)$, let $x=-3$, then $f(-3)=$ ______________.
- For $(-2,-1)$, let $x=-1.5$, then $f(-1.5)=$ ______________.
- For $(-1,1)$, let $x=0$, then $f(0)=$ ______________.
- For $(2,\infty)$, let $x=3$, then $f(3)=$ ______________.

Therefore, the solution set is ____________ $\cup$ ____________.

Remark

In the previous examples, note that we determined the sign of $f(x)$ instead of the value of $f(x)$. This is because we are only interested in whether $f(x)$ is positive, negative, or zero to solve the inequality.

Exercises

Exercise 2.8.1. Solve the inequality $-x^2 > 5x - 6$.

Answer: $(-6, 1)$.

Exercise 2.8.2. Solve the inequality $2x^3 + x^2 \leq 2x + 1$.

Answer: $(-\infty, -1] \cup \left[\frac{1}{2}, \infty\right)$.

Exercise 2.8.3. Solve the inequality $1 \geq \dfrac{x-1}{2x+1}$.

Answer: $(-\infty, -2] \cup (-\frac{1}{2}, \infty)$.

Exercise 2.8.4. Solve the inequality $\dfrac{x+8}{x^2-4} < 1$.

Answer: $(-\infty, -3) \cup (-2, 2) \cup (4, \infty)$.

Chapter 3 Exponential and Logarithmic Functions

3.1 Exponential Functions

Definition 3.1.1 (Exponential Functions)

For any real number x (the exponent), an **exponential function** f of x is a function defined by an equation

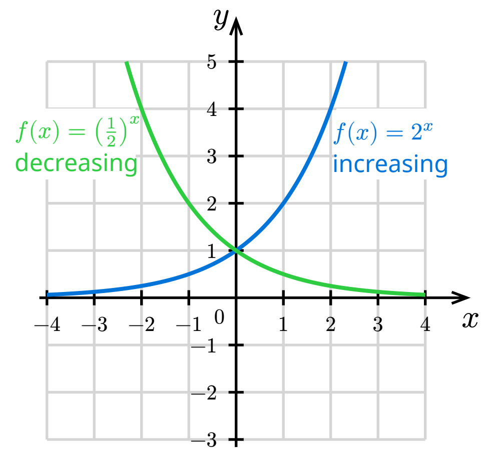
$$f(x) = b^x,$$

where b is a positive real number, called the **base**, such that $b \neq 1$.

Properties of Exponential Functions

Consider the exponential function $f(x) = b^x$, where $b > 0$, and $b \neq 1$.

- The *domain* of f is $(-\infty, \infty)$, and the range of f is $(0, \infty)$.
- The *y-intercept* of f is $(0, 1)$, and the function has a *horizontal asymptote* $y = 0$.
- The function f is increasing if $b > 1$ or decreasing if $0 < b < 1$.



Example 3.1.1. The population of India was about 1.25 billion in the year 2013, with an annual growth rate of about 1.2%. This situation is represented by the growth function $P(t) = 1.25(1.012)^t$, where t is the number of years since 2013. To the nearest thousandth, what will the population of India be in 2031?

Solution. In the year 2031, $t = 2031 - 2013 =$ ______. To find the population after 18 years, evaluate P at $t = 18$:

$$P(18) = 1.25(1.012)^{\_\_\_\_} \approx 1.518 \text{ billion.}$$

Thus, to the nearest thousandth, the population of India in 2031 is approximately 1.518 billion.

Remark

When modeling real-world situations with exponential functions, an initial value factor is often included in the function. A common general form used in applications is:

$$f(t) = ab^t,$$

where a represents the initial value and b is the base of the exponential growth or decay. If $b > 1$, the function models exponential growth; if $0 < b < 1$, it models exponential decay.

Example 3.1.2. In 2006, 80 deer were introduced into a wildlife refuge. By 2012, the population had grown to 180 deer. The population was growing exponentially. Write an algebraic function $N(t)$ representing the population N of deer over time t.

Solution. Since the population growing exponentially, we can model the population by the function

$$N(t) = ab^t,$$

where a is the initial population, and b is the growth factor. Since the initial population is 80 deer, we take $a = 80$. Thus,

$$N(t) = 80b^t.$$

To find b, we use the information that after ______ years (from 2006 to 2012), the population is 180 deer. Thus,

$$180 = N(6) = 80b^6.$$

Solving for b, we get

$$b^6 = \frac{180}{80}$$

$$b = \left(\frac{9}{4}\right)^{\_\_\_\_}$$

$$b \approx 1.144.$$

Therefore, the population of deer over time t is modeled by the function

$$N(t) = \_\_\_\_\_\_\_\_ .$$

Example 3.1.3. Find an exponential function $f(x) = ab^x$ that passes through the points $(-2, 6)$ and $(2, 1)$. Round to three decimal places.

Solution. From the given points, we have the system of equations

$$\begin{cases} 6 = ab^{-2} \\ 1 = ab^2 \end{cases}$$

Dividing the first equation by the second and solving for b gives:

$$\frac{6}{1} = \frac{ab^{-2}}{ab^2}$$

$$6 = \_\_\_\_\_\_$$

$$b^4 = \frac{1}{6}$$

$$b = \left(\frac{1}{6}\right)^{\_\_\_\_} \approx 0.638.$$

Multiplying the first equation with the second and solving for a gives:

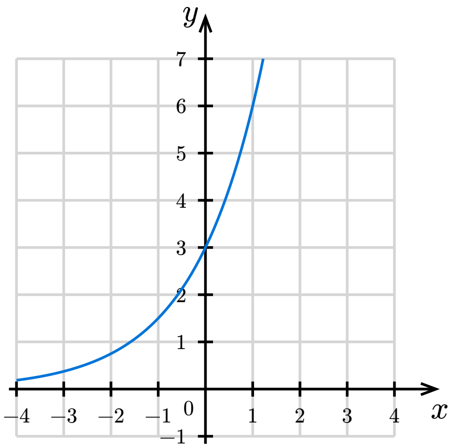
$$6 = a^2$$

$$a = \_\_\_\_\_\_ \approx 2.449.$$

Thus, an exponential function that passes through the given points is

$$f(x) = \_\_\_\_\_\_\_\_ .$$

Example 3.1.4. Find an exponential function $f(x) = ab^x$ graphed in the following figure.



Solution. Two points are needed to find the values of a and b. Since the y-intercept is $(0, -3)$, the initial value is

$$a = \underline{\qquad}.$$

On the graph, there is another point $(1, -6)$. Substituting into the equation and solving for x gives:

$$-6 = -3b^1$$

$$b = \frac{-6}{-3} = \underline{\qquad}.$$

Therefore, an equation of the graphed function is

$$f(x) = \underline{\qquad\qquad}.$$

Definition 3.1.2 (The Number e)

The natrual number, denoted by e, the number that $\left(1 + \frac{1}{n}\right)^n$ approaches to as n increases without bound. Approximately, $e \approx 2.718282$. ♣

Example 3.1.5. Evaluate using a calculator. Round to five decimal places.

1) e^2 2) $e^{-\frac{1}{2}}$ 3) e^{π}

Solution.

1) $e^2 \approx \underline{\qquad\qquad}$ 2) $e^{-\frac{1}{2}} \approx \underline{\qquad\qquad}$ 3) $e^{\pi} \approx \underline{\qquad\qquad}$

Investment Models

Let P be the initial amount of the account, known as the principal, r the annual interest rate, and t is the number of years. The balance A after t years is

- $A(t) = P\left(1 + \frac{r}{n}\right)^{nt}$ if the interest is compounded n times per year.
- $A(t) = Pe^{rt}$ if the interest is compounded continuously ($n \to \infty$).

Example 3.1.6. If $\$3,000$ is invested in a savings account paying 3% interest compounded quarterly, how much will the account be worth in 10 years?

Solution. Here, $P = \underline{\qquad}$, $r = \underline{\qquad}$, $n = \underline{\qquad}$, and $t = \underline{\qquad}$. Using the formula for compound interest, we have

$$A(10) = 3000\left(1 + \frac{0.03}{4}\right)^{4 \cdot 10} \approx \underline{\qquad\qquad\qquad}.$$

Thus, the account will be worth approximately $ $\underline{\qquad\qquad}$ in 10 years.

Example 3.1.7. A person invested $\$1,000$ in an account earning 10% per year compounded continuously. How much was in the account at the end of two and a half year?

Solution. Here, $P =$ ______, $r =$ ______, and $t =$ ______. Using the formula for continuous compounding, we have

$$A(2.5) = 1000e^{0.10 \cdot 2.5} \approx \underline{\qquad\qquad} .$$

Thus, the account will be worth approximately $\$$ __________ at the end of two and a half years.

Example 3.1.8. A 529 Plan is a college-savings plan that allows relatives to invest money to pay for a child's future college tuition; the account grows tax-free. Lily wants to set up a 529 account for her new granddaughter and wants the account to grow to $\$40,000$ over 18 years. She believes the account will earn 6% compounded semi-annually (twice a year). To the nearest dollar, how much will Lily need to invest in the account now?

Solution. Here, $t = 10$, $A(10) =$ __________, $r =$ ______, and $n =$ ______. Using the formula for compound interest, we have

$$40000 = P\left(1 + \frac{0.06}{2}\right)^{2 \cdot 18}.$$

Solving for P, we get

$$P = \frac{40000}{(1 + \frac{0.06}{2})^{36}} \approx \underline{\qquad\qquad} .$$

Thus, Lily will need to invest approximately $\$$ __________ in the account now.

Continuous Growth/Decay Model

When modeling continuous growth or decay, the number e is usually used as the base of the exponential function: $A(t) = A_0 e^{kt}$, where A_0 is the initial amount and k is the continuous growth rate (if $k > 0$) or decay rate (if $k < 0$), expressed as a decimal, t is the time and $A(t)$ is the amount after time t.

Example 3.1.9. *Radon*-222 decays at a *continuous* rate of 17.3% per day. How much will $100mg$ of *Radon*-222 decay to in 3 days? Round to the nearest hundredth.

Solution. The decay of *Radon*-222 can be modeled by the function

$$A(t) = A_0 e^{kt},$$

where $A(t)$ is the amount remaining after time t, $A_0 =$ ________ is the initial amount, and $k =$ ________ is the continuous rate of decay (as a negative decimal).

Therefore, the remaining amount after 3 days is given by

$$A(3) = 100e^{-0.173 \cdot 3} \approx \underline{\qquad\qquad} .$$

After 3 days, approximately 59.51 mg of *Radon*-222 will remain.

Exercises

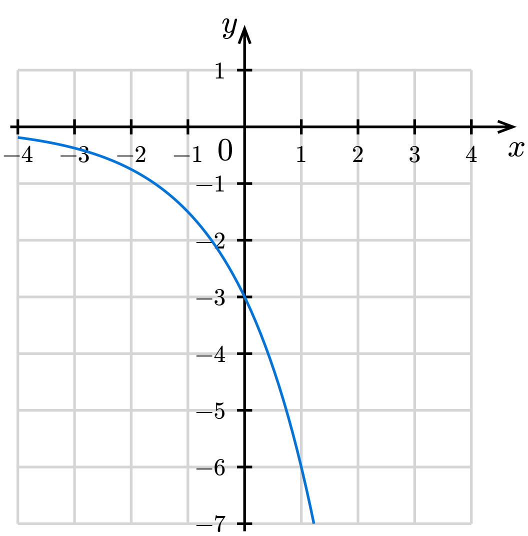
Exercise 3.1.1. A vehicle depriciates according to the formula: $v = 27500(3.42)^{-.04x}$ where x is the age of the car in years. Find the value of the car when it is 14-years old.

Answer: $v = 27500(3.42)^{-.04 \cdot 14} \approx \$13812.684.$

Exercise 3.1.2. Find an exponential function $f(x) = ab^x$ that passes through the points $(-2, -6)$ and $(-1, -3)$.

Answer: $f(x) = -\frac{3}{2}\left(\frac{1}{2}\right)^x = -3\left(\frac{1}{2}\right)^{x+1}.$

Exercise 3.1.3. Find an exponential function $f(x) = ab^x$ graphed in the following figure.



Answer: $f(x) = -3 \cdot 2^x.$

Exercise 3.1.4. A wolf population is growing exponentially. In 2011, 129 wolves were counted. By 2013, the population had reached 236 wolves. What two points can be used to derive an exponential equation modeling this situation? Write the equation representing the population N of wolves over time t.

Answer: $N(t) = 129\left(\sqrt{\frac{236}{129}}\right)^t \approx 129(1.353)^t$.

Exercise 3.1.5. A scientist begins with 100 milligrams of a radioactive substance that decays exponentially. After 35 hours, 50 mg of the substance remains. How many milligrams will remain after 54 hours?

Answer: $A(54) = 100\left(\frac{1}{2}\right)^{\frac{54}{35}} \approx 34.321$ mg.

Exercise 3.1.6. An account is opened with an initial deposit of $\$6,500$ and earns 3.6% interest.

1) What will the account be worth in 20 years if the interest is compounded monthly.

2) What will the account be worth in 20 years if the interest is compounded continuously.

Answer: 1) $A(20) = 6500\left(1 + \frac{0.036}{12}\right)^{12\cdot 20} \approx \$13339.43.$ 2) $A(20) = 6500e^{0.036\cdot 20} \approx \$13353.82.$

3.2 Logarithmic Functions

Definition 3.2.1 (Logarithmic Functions)

Let $y = b^x$ be an exponential function, where $b > 0$ and $b \neq 1$. Its inverse is called the **logarithmic function with base b**, written as: $y = \log_b x$.

The notation $\log_b x$ is read as "log base b of x." The value of $\log_b x$ is called the **logarithm of x to the base b**, and x is called the **argument** of the logarithm. ♣

Basic Properties of a Logarithmic Function

For $b > 0$ and $b \neq 1$, as the functions $y = b^x$ and $y = \log_b x$ are inverses of each other, in particular, $y = b^x$ and $x = \log_b y$ are equivalent equations. Moreover,

1) $b^{\log_b x} = x$ for $x > 0$.

2) $\log_b(b^x) = x$ for all real numbers x, in particular,

$$\log_b b = 1 \qquad \text{and} \qquad \log_b 1 = 0.$$

Example 3.2.1. Write the following logarithmic equality in exponential form.

1) $\log_2(x) = 3$

2) $\log_x(5) = \frac{1}{3}$

Solution.

1) ________ $= x$

2) ________ $= 5$

Example 3.2.2. Use the exponential form to evaluate the logarithm.

1) $\log_2 4$

2) $\log_2 \sqrt{2}$

3) $\log_9 3$

4) $\log_5\left(\frac{1}{25}\right)$

Solution.

1) $\log_2 4 = \log_2 2^2 =$ ______

2) $\log_2 \sqrt{2} = \log_2 2^{\_\_\_} =$ ______

3) $\log_9 3 = \log_9 9^{\_\_\_} =$ ______

4) $\log_5\left(\frac{1}{25}\right) = \log_5 5^{\_\_\_} =$ ______

Definition 3.2.2 (Common and Natural Logarithms)

A **common logarithm** is a logarithm with base 10. We write $\log_{10}(x)$ simply as $\log(x)$.

A **natural logarithm** is a logarithm with base e, the natrual number. We write $\log_e(x)$ simply as $\ln(x)$. ♣

Example 3.2.3. Evaluate the logarithm **without using a calculator**.

1) $\log(1000)$

2) $\ln(e^2)$

Solution.

1) $\log(1000) = \log(10^{\_\_\_\_\_}) =$ ________

2) $\ln(e^2) =$ ______

Example 3.2.4. Evaluate the logarithm using a calculator.

1) $\log 2$

2) $\ln 2$

Solution.

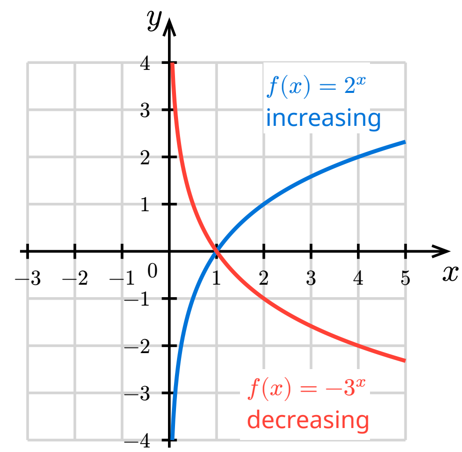
1) $\log 2 \approx$ ________

2) $\ln 2 \approx$ ________

Domains and Ranges of Logarithmic Functions

Consider the logarithmic function $f(x) = \log_b x$, where $b > 0$ and $b \neq 1$.

- The *domain* of f is $(0, \infty)$, and its *range* is $(-\infty, \infty)$.
- The x-*intercept* is $(1, 0)$, and the function has a *vertical asymptote* at $x = 0$. Note that 0 is the finite boundary of the domain.
- The function f is increasing if $b > 1$ and decreasing if $0 < b < 1$.



Example 3.2.5. Find the domain of each function:

1) $f(x) = \log_3(3 - 2x)$

2) $f(x) = \log\left(\frac{x+1}{x-2}\right)$

Solution.

1) For $f(x) = \log_3(3 - 2x)$, the domain is determined by:

$$3 - 2x > 0.$$

Solving the inequality gives: $x <$ ______.

So the domain is: $(-\infty,$ ______$)$.

2) For $f(x) = \log\left(\frac{x+1}{x-2}\right)$, the domain is determined by:

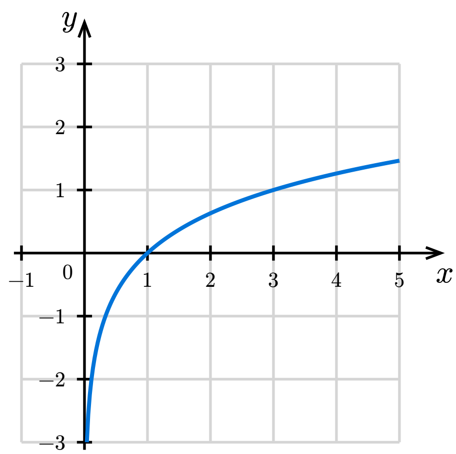
$$\frac{x+1}{x-2} > 0.$$

Solve using the test-point method:

- Critical points: $x = -1$ and $x = 2$, giving intervals $(-\infty, -1)$, $(-1, 2)$, and $(2, \infty)$.
- Testing each interval shows that the inequality holds over $($______$, -1) \cup (2,$ ______$)$.

Therefore, the domain of f is $(-\infty,$ ______$) \cup ($______$, \infty)$.

Example 3.2.6. Find an equation for the function $y = \log_b x$ whose graph is shown below.



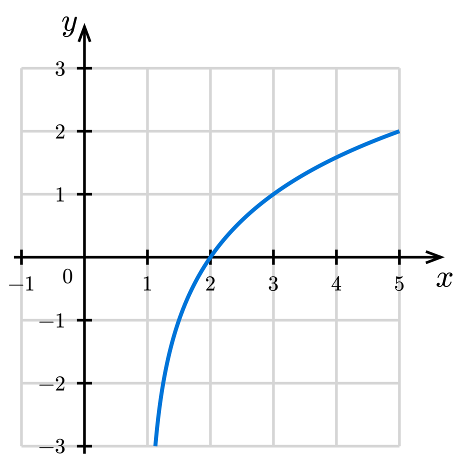
Solution. Since the logarithmic function has the form $y = \log_b x$, we need to determine the base b. Use another point on the graph, for example $(3, 1)$.

Substituting into the equation and solving for b gives:

$$1 = \log_b 3$$
$$b = 3^1 = \_\_\_\_\_.$$

Therefore, the equation of the function is: $y =$ __________.

Example 3.2.7. Find an equation for the function $y = \log_b(x - a)$ whose graph is shown below.



Solution. Two points are needed to determine the base b and the horizontal shift a. From the graph, the x-intercept is $(2, 0)$, so:

$$2 - a = 1$$
$$a = \_\_\_\_\_.$$

On the graph there is another point $(3, 1)$. Substituting into the equation and solving for b:

$$1 = \log_b(3 - 1)$$
$$b = \_\_\_\_\_.$$

Therefore, the equation of the function is:

$$y = \_\_\_\_\_\_\_\_\_\_.$$

Example 3.2.8. Find the x-intercept and the vertical asymptote of $f(x) = -\log_3(x + 4)$

Solution. The function is obtained by applying a horizontal shift and a vertical reflection to the basic logarithmic function $y = \log_3 x$. A vertical reflection preserves vertical lines and the x-axis, so the x-intercept and vertical asymptote shift accordingly.

Since the x-intercept of $y = \log_3 x$ is $(1, 0)$ and its vertical asymptote is $x = 0$, the x-intercept and vertical asymptote of $f(x) = -\log_3(x + 4)$ are determined by:

$$x + 4 = 1 \qquad\qquad x + 4 = 0$$
$$x = \_\_\_\_\_ \qquad\qquad x = \_\_\_\_\_$$

Therefore, the x-intercept is $(\_\_\_\_\_, 0)$ and the vertical asymptote is $x = \_\_\_\_\_$.

Exercises

Exercise 3.2.1. Write the following logarithmic equality in exponential form.

1) $\log_4 2 = x$ 2) $\log_3(x) = 2$ 3) $\log_x(2) = \frac{1}{2}$

Answer: 1) $4^x = 2$ 2) $3^2 = x$ 3) $x^{\frac{1}{2}} = 2$

Exercise 3.2.2. Evaluate the logarithm using a calculator.

1) $\log 3$ 2) $\ln 5$ 3) $\frac{\log 5}{\ln 3}$

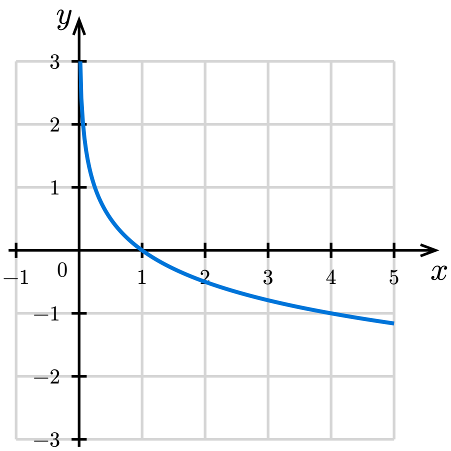
Answer: 1) $\log 3 \approx 0.47712$ 2) $\ln 5 \approx 1.60944$ 3) $\frac{\log 5}{\ln 3} \approx 0.63623$

Exercise 3.2.3. Find the domain of the function.

1) $f(x) = \log_2(2x - 1)$ 2) $f(x) = \ln(9 - 4x^2)$ 3) $f(x) = \log\left(\frac{x-1}{x-2}\right)$

Answer: 1) Domain: $\left(\frac{1}{2}, \infty\right)$ 2) Domain: $\left(-\frac{3}{2}, \frac{3}{2}\right)$ 3) Domain: $(-\infty, 1) \cup (2, \infty)$

Exercise 3.2.4. Find an equation for the function $y = -\log_b x$ whose graph is shown below.



Answer: $y = -\log_4 x$.

Exercise 3.2.5. Find the vertical asymptote of $f(x) = -3\log_2(2x-1)+1$

Answer: $x = \frac{1}{2}$.

3.3 Review of Properties of Logarithms

Properties of Logarithms

Assume $M > 0$, $N > 0$, $b > 0$ and $b \neq 1$. Then

Product Rule:

$$\log_b(MN) = \log_b M + \log_b N.$$

Quotient Rule:

$$\log_b\left(\frac{M}{N}\right) = \log_b M - \log_b N.$$

Power Rule:

$$\log_b(M^p) = p\log_b M,$$

where p is any real number.

Change-of-base Property:

$$\log_b M = \frac{\log_a M}{\log_a b},$$

where $a > 0$ and $a \neq 1$. In particular,

$$\log_b M = \frac{\log M}{\log b} \quad \text{and} \quad \log_b M = \frac{\ln M}{\ln b}.$$

Example 3.3.1. Expand the logarithmic expression.

1) $\log_3(30x(3x+4))$ 2) $\log\left(\sqrt{x^2+1}\right)$ 3) $\ln\left(\frac{x^4(y-1)}{x^2+1}\right)$

Solution.

1)

$$\begin{aligned}\log_3(30x(3x+4)) &= \log_3 30 + \log_3 x + \log_3(\underline{\hspace{3cm}}) \\ &= \log_3 3 + \log_3 10 + \log_3 x + \log_3 3 + \log_3(x+4) \\ &= \underline{\hspace{1.5cm}} + \log_3 10 + \log_3 x + \log_3(x+4).\end{aligned}$$

2)

$$\begin{aligned}\log\left(\sqrt{x^2+1}\right) &= \log\left(x^2+1\right)^{\underline{\hspace{1.5cm}}} \\ &= \underline{\hspace{1.5cm}}\log(x^2+1).\end{aligned}$$

3)

$$\begin{aligned}\ln\left(\frac{x^4(y-1)}{x^2+1}\right) &= \ln\left(x^4(\underline{\hspace{2cm}})\right) - \ln(\underline{\hspace{2cm}}) \\ &= \ln(x^4) + \ln(y-1) - \ln(x^2+1) \\ &= \underline{\hspace{1.5cm}}\ln(x) + \ln(y-1) - \ln(x^2+1).\end{aligned}$$

Example 3.3.2. Condense the logarithmic expression.

1) $\log_2(x^2) + \frac{1}{2}\log_2(x-1) - 3\log_2((x+3)^2)$ **2)** $3\ln(x) - \frac{1}{2}\ln(x+1) - 2\ln\left(\sqrt{x^2+3}\right)$

Solution.

1)

$$\begin{aligned} &\log_2(x^2) + \frac{1}{2}\log_2(x-1) - 3\log_2((x+3)^2) \\ &= \log_2(x^2) + \log_2(x-1)^{\underline{\qquad}} - \log_2((x+3)^{2\cdot 3}) \\ &= \log_2 \frac{x^2(\underline{\qquad\qquad})}{\underline{\qquad\qquad}} \\ &= \log_2 \frac{x^2\sqrt{x-1}}{(x+3)^6}. \end{aligned}$$

2)

$$\begin{aligned} &3\ln(x) - \frac{1}{2}\ln(x+1) - 2\ln\left(\sqrt{x^2+3}\right) \\ &= \ln(x^3) - \ln(x+1)^{\underline{\qquad}} - \ln\left(\left(\sqrt{x^2+3}\right)^{\underline{\qquad}}\right) \\ &= \ln(x^3) - \ln\left(\sqrt{x+1}\right) - \ln(\underline{\qquad\qquad}) \\ &= \ln \frac{x^3}{\sqrt{(x+1)(x^2+3)}}. \end{aligned}$$

Exercises

Exercise 3.3.1. Expand the logarithmic expression.

1) $\log_6\left(\dfrac{64x^3(4x+1)}{(2x-1)}\right)$

2) $\ln\left(\dfrac{\sqrt{(x-1)}(2x+1)^2}{(x^2-9)}\right)$

Answer: **1)** $6\log_6 2 + 3\log_6 x + \log_6(4x+1) - \log_6(2x-1)$. **2)** $\frac{1}{2}\ln(x-1) + 2\ln(2x+1) - \ln(x-3) - \ln(x+3)$.

Exercise 3.3.2. Condense the logarithmic expressions.

1) $3\log(x) + \log(x+5) - \log(2x+3)$

2) $2\log x - 4\log(x+5) + \frac{1}{3}\log(3x+5)$

Answer: **1)** $\log\left(\frac{x^2(x+5)}{2x+3}\right)$. **2)** $\log\left(\frac{x^2\sqrt[3]{3x+5}}{(x+5)^4}\right)$.

3.4 Exponential and Logarithmic Equations

Guidelines to Solve Exponential Equations

1) **Isolate** the exponential expression on one side of the equation.
2) **Take the logarithm** of both sides—preferably with the same base as the exponential, but any consistent base (e.g., common log or natural log) may be used.
3) **Solve** the resulting algebraic equation.

Example 3.4.1. Solve

1) $3^{x+1} = 4$ 2) $2^{x-1} = 4^{x-2}$ 3) $5^{x+2} = 4^x$

Solution.

1)
$$\begin{aligned} 3^{x+1} &= 4 \\ \underline{\qquad\qquad} &= \log_3 4 \\ x &= \log_3 4 - 1. \end{aligned}$$

2)
$$\begin{aligned} 2^{x-1} &= 4^{x-2} \\ x-1 &= (x-2)\underline{\qquad\quad} \\ x-1 &= 2(x-2) \\ x-1 &= 2x + \underline{\qquad\quad} \\ x &= \underline{\qquad\quad}. \end{aligned}$$

3)
$$\begin{aligned} 5^{x+2} &= 4^x \\ \ln 5(x+2) &= x\ \underline{\qquad\quad} \\ x\ln 5 + \underline{\qquad\quad} &= x\ln 4 \\ x(\underline{\qquad\quad}) &= -2\ln 5 \\ x &= \frac{-2\ln 5}{\ln 5 - \ln 4}. \end{aligned}$$

Example 3.4.2. Solve

1) $100 = 20e^{2t}$ 2) $4e^{2x} + 5 = 12$ 3) $e^{2x} - e^x = 56$

Solution.

1)
$$\begin{aligned} 100 &= 20e^{2t} \\ \underline{\qquad\quad} &= e^{2t} \\ \ln 5 &= \underline{\qquad\quad} \\ t &= \frac{\ln 5}{2}. \end{aligned}$$

2)
$$\begin{aligned} 4e^{2x} + 5 &= 12 \\ 4e^{2x} &= \underline{\qquad\quad} \\ e^{2x} &= \frac{7}{4} \\ \underline{\qquad\quad} &= \ln\left(\frac{7}{4}\right) \\ x &= \frac{1}{2}\ln\left(\frac{7}{4}\right). \end{aligned}$$

3)
$$\begin{aligned} e^{2x} - e^x &= 56 \\ (e^x)^2 - e^x - 56 &= 0 \\ (e^x - 8)(e^x + \underline{\qquad\quad}) &= 0 \\ e^x - 8 = 0 &\text{ or } e^x + 7 = 0 \\ e^x &= \underline{\qquad\quad} \text{ or no solution} \\ x &= \underline{\qquad\quad}. \end{aligned}$$

Guidelines to Solve Logarithmic Equations

1) **Isolate** the logarithmic expression (use log properties if needed).
2) **Exponentiate** both sides using the logarithm's base.
3) **Solve** the resulting equation.
4) **Check** solutions in the original equation—discard any outside the domain of the logarithms.

Example 3.4.3. Solve

1) $2\ln x + 3 = 7$ **2)** $\ln(x^2) = \ln(2x+3)$ **3)** $\ln(x) - \ln(x+3) = \ln 6$

Solution.

1)

$$\begin{aligned} 2\ln x + 3 &= 7 \\ 2\ln x &= \underline{\qquad} \\ \ln x &= \underline{\qquad} \\ x &= \underline{\qquad}. \end{aligned}$$

Check: Because $x = e^2 > 0$ is in the domain of the original equation. So, $x = e^2$ is a solution.

2)

$$\begin{aligned} \ln(x^2) &= \ln(2x+3) \\ x^2 &= 2x+3 \\ x^2 - 2x - 3 &= 0 \\ (\underline{\qquad\quad})(x+1) &= 0 \\ x - 3 = 0 \quad &\text{or} \quad x + 1 = 0 \\ x = \underline{\qquad} \quad &\text{or} \quad x = \underline{\qquad} \end{aligned}$$

When $x = 3$, both $x^2 = 9 > 0$ and $2x + 3 = 9 > 0$. So, $x = 3$ is a solution.
When $x = -1$, both $x^2 = 1$ and $2x + 3 = \underline{\qquad} > 0$. So, $x = -1$ is also a solution.

3)

$$\begin{aligned} \ln(x) - \ln(x+3) &= \ln 6 \\ \ln\left(\frac{x}{x+3}\right) &= \ln 6 \\ \frac{x}{x+3} &= 6 \\ x &= 6(x+3) \\ -5x &= \underline{\qquad} \\ x &= \underline{\qquad}. \end{aligned}$$

Check: Since $x = -\frac{18}{5} < 0$ is not in the domain of $\ln x$, it is an extraneous solution and must be discarded. The equation has **no solution**.

Example 3.4.4. An account with an initial deposit of $\$6,500$ earns 7.25% annual interest, compounded monthly. After how many years, the balance will be doubled. Round your answer to the nearest hundredth.

Solution. Let $A(t)$ be the amount in the account after t years. Using the compound interest formula:

$$A(t) = 6500\left(1 + \frac{0.0725}{12}\right)^{12t}.$$

The number of years it takes to double the balance satisfies:

$$\underline{\qquad} = 6500\left(1 + \frac{0.0725}{12}\right)^{12t}.$$

Divide by 6500 and solve for t:

$$2 = \left(1 + \frac{0.0725}{12}\right)^{12t}$$

$$(\underline{\qquad})\log\left(1 + \frac{0.0725}{12}\right) = \log 2$$

$$t = \underline{\qquad\qquad} \approx 9.59.$$

Therefore, it will take approximately 9.59 years for the balance to double.

Example 3.4.5. The magnitude M of an earthquake is represented by the equation

$$M = \frac{2}{3}\log\left(\frac{E}{E_0}\right),$$

where E is the amount of energy released by the earthquake in joules, and $E_0 = 10^{4.8}$ is the assigned minimal measure released by an earthquake. To the nearest hundredth, if the magnitude of an earthquake is 7.8, how much energy was released? Answer with exact value in scientific notation.

Solution. Substituting $M = 7.8$ into the equation gives:

$$7.8 = \frac{2}{3}\log\left(\frac{E}{10^{4.8}}\right).$$

Solve for E and simplify the answer:

$$\log\left(\frac{E}{10^{4.8}}\right) = \underline{\qquad}$$

$$\frac{E}{10^{4.8}} = 10^{\underline{\qquad}}$$

$$E = 10^{\underline{\qquad}}.$$

Therefore, the amount of energy released by the earthquake is approximately $10^{16.5}$ joules.

Exercises

Exercise 3.4.1. Solve

1) $3^{1-x} = 5$ 2) $3^{x-2} = 4^{2x}$ 3) $5 = 10^{3t-2}$ 4) $e^{2x} - 2e^{x} = 15$

Answer: 1) $x = 1 - \log_3 5$ 2) $x = \frac{4\log 2 + 2}{\log 3 + 2\log 4}$ 3) $t = \frac{\log 5 + 2}{3\log 10}$ 4) $x = \ln 5$

Exercise 3.4.2. Solve

1) $2\log x - 3 = -1$

2) $\ln(2x^2) = \ln(5x+3)$

3) $\frac{1}{2}\log_2(3x-1) = 2$

4) $\ln(x-1) - \ln(x+1) = 1$

Answer: 1) $x = 10$ 2) $x = 3$ 3) $x = 5$ 4) no solution

Exercise 3.4.3. An account with an initial deposit of $\$8,200$ earns 6.4% annual interest, compounded quarterly. After how many years will the balance be tripled? Round your answer to the nearest hundredth.

Answer: Approximately 17.3 years.

Exercise 3.4.4. The loudness L of a sound in decibels is given by

$$L = 10\log\left(\frac{I}{I_0}\right),$$

where I is the sound intensity (in watts per square meter) and $I_0 = 10^{-12}$ is the reference intensity.

To the nearest hundredth, if a sound has a loudness of 95 decibels, what is its intensity I? Give the exact value in scientific notation.

Answer: $10^{-2.5}$ watts per square meter.

3.5 Exponential and Logarithmic Models

Exponential Growth and Decay

Recall that the function

$$A(t) = A_0 e^{kt} \quad \text{or equivalently} \quad A(t) = A_0 b^t$$

is commonly used to model exponential growth (when $k > 0$ or $b > 1$) or decay (when $k < 0$ or $0 < b < 1$), where A_0 is the initial quantity.

Example 3.5.1. A population of bacteria doubles every hour. A culture started with 10 bacteria.

1) After 6 hours how many bacteria will there be?
2) After how many hours will the population be tripled? Round your answer to the nearest hundredth.

Solution. The initial population is 10 bacteria, so the population after t hours can be modeled by:

$$P(t) = 10b^t.$$

Since the population doubles every hour:

$$b = \frac{P(1)}{P(0)} = \_\_\_\_\_\_.$$

Thus, the population after t hours is:

$$P(t) = 10(2)^t.$$

1) After 6 hours:

$$P(6) = 10(2)^6 = \_\_\_\_\_\_\_\_\_\_.$$

2) The time t needed to triple the population satisfies:

$$\_\_\_\_\_\_ = 102^t.$$

Solve for t:

$$\begin{aligned} 3 &= 2^t \\ \log 3 &= \_\_\_\_\_\_\_\_ \\ t &= \_\_\_\_\_\_ \approx 1.58. \end{aligned}$$

Therefore, it will take approximately 1.58 hours for the population to triple.

Example 3.5.2. The half-life of carbon-14 is $5,730$ years. Laboratory analysis shows that a bone fragment currently contains 20% of the carbon-14 that a living organism would have had. Estimate the age of the bone to the nearest year.

Solution. Let $A(t)$ be the amount of carbon-14 remaining after t years. By the exponential decay formula:

$$A(t) = A_0 e^{kt}.$$

Since the half-life of carbon-14 is $5,730$ years, k satisfies

$$\frac{1}{2}A_0 = A_0 e^{5730k}.$$

Divide by A_0 and solve for k:

$$k = \frac{\underline{\hspace{2cm}}}{5730}.$$

Thus, the model is:

$$A(t) = A_0 e^{\frac{\ln\left(\frac{1}{2}\right)}{5730}t} = A_0 \left(e^{\ln\left(\frac{1}{2}\right)}\right)^{\frac{5730}{t}} = A_0 \left(\frac{1}{2}\right)^{\frac{t}{5730}}.$$

The age t of the bone fragment when 20% of the original carbon-14 remains satisfies:

$$\underline{\hspace{2cm}}\ A_0 = A_0 \left(\frac{1}{2}\right)^{\frac{t}{5730}}.$$

Divide by A_0 and solve for t:

$$0.2 = \left(\frac{1}{2}\right)^{\frac{t}{5730}}$$

$$\left(\frac{\ln\left(\frac{1}{2}\right)}{5730}\right)t = \underline{\hspace{2cm}}$$

$$t = \underline{\hspace{4cm}} \approx 13305.$$

Therefore, the bone fragment is approximately 13305 years old.

Remark

In the previous example, we could have also used the formula

$$A(t) = A_0 \left(\frac{1}{2}\right)^{\frac{t}{h}},$$

where h is the half-life of the substance.

Example 3.5.3. Sam goes to the doctor and the doctor gives him 15 milligrams of radioactive dye. After 15 minutes, 9 milligrams of dye remain in Sam body. To leave the doctor's office, Sam must pass through a radiation detector that will sound the alarm if more than 2 milligrams of the dye are in his body. How long Sam's visit to the doctor take, assuming he was given the dye as soon as he arrived? Round your answer to the nearest minute.

Solution. Let $A(t)$ be the amount of dye remaining in Sam's body after t minutes. By the exponential decay formula:

$$A(t) = \underline{\hspace{2cm}}\ e^{kt}.$$

Since 9 milligrams remain after 15 minutes, the decay constant k satisfies:

$$9 = 15e^{15k}.$$

Solve for k:

$$e^{15k} = \frac{3}{5}$$
$$\underline{\qquad} = \ln\left(\frac{3}{5}\right)$$
$$k = \underline{\qquad}.$$

Thus, the model is:

$$A(t) = 15e^{\frac{\ln\left(\frac{3}{5}\right)}{15}t}.$$

The time that Sam's visit will take satisfies:

$$\underline{\qquad} = 15e^{\frac{\ln\left(\frac{3}{5}\right)}{15}t}.$$

Solve for t again:

$$\frac{2}{15} = e^{\frac{\ln\left(\frac{3}{5}\right)}{15}t}$$
$$\left(\frac{\ln\left(\frac{3}{5}\right)}{15}\right)t = \underline{\qquad}$$
$$t = \underline{\qquad\qquad} \approx 59.17.$$

Therefore, Sam's visit will take approximately 59.17 minutes.

Newton's Law of Cooling

The temperature of an object, T, in surrounding air with constant temperature T_s, will behave according to the formula

$$T(t) = Ae^{kt} + T_s,$$

where t is time, A is the difference between the initial temperature of the object and the surroundings, k is a constant, the continuous rate of cooling of the object.

Example 3.5.4. A cheesecake is taken out of the oven with an ideal internal temperature of $165°\mathrm{F}$, and is placed into a $35°\mathrm{F}$ refrigerator. After 10 minutes, the cheesecake has cooled to $150°\mathrm{F}$. If we must wait until the cheesecake has cooled to $70°\mathrm{F}$ before we eat it, how long will we have to wait?

Solution.

Let $T(t)$ be the temperature of the cheesecake after t minutes. By Newton's Law of Cooling:

$$T(t) = Ae^{kt} + T_s,$$

where $T_s = 35°\mathrm{F}$ is the surrounding temperature and $A = 165 - 35 = \underline{\qquad}$.

Thus, the temperature after t minutes is:

$$T(t) = 130e^{kt} + 35.$$

Since the temperature drops to $150°\mathrm{F}$ after 10 minute, the constant k satisfies:

$$150 = 130e^{10k} + 35.$$

Solve for k:

$$\begin{aligned} 115 &= 130e^{10k} \\ e^{10k} &= \frac{23}{26} \\ \underline{\qquad\quad} &= \ln\left(\frac{23}{26}\right) \\ k &= \underline{\qquad\quad}. \end{aligned}$$

So the temperature after t minutes is:

$$T(t) = 130e^{\frac{\ln\left(\frac{23}{26}\right)}{10}t} + 35.$$

The time it takes the cheesecake to cool to $70°\mathrm{F}$ satisfies:

$$70 = 130e^{\frac{\ln\left(\frac{23}{26}\right)}{10}t} + 35.$$

Solve for t:

$$\begin{aligned} \underline{\qquad\quad} &= 130e^{\frac{\ln\left(\frac{23}{26}\right)}{10}t} \\ e^{\frac{\ln\left(\frac{23}{26}\right)}{10}t} &= \frac{7}{26} \\ \left(\underline{\qquad\qquad\qquad}\right)t &= \ln\left(\frac{7}{26}\right) \\ t &= \underline{\qquad\qquad\qquad} \approx 107.03. \end{aligned}$$

Therefore, we must wait approximately 107.03 minutes for the cheesecake to cool to $70°\mathrm{F}$.

Logistic Growth

The logistic growth model behaves approximately exponentially at first, but its growth rate decreases as the population approaches an upper limit called the *carrying capacity*.

The population at time t is modeled by

$$P(t) = \frac{c}{1 + ae^{-bt}},$$

where a, b, and c are positive constants with the following interpretations:

- c is the carrying capacity—the value that $P(t)$ approaches as $t \to \infty$;
- b is the growth rate;
- a is determined by the initial population, specifically
$$a = \frac{c - P(0)}{P(0)}.$$

Example 3.5.5. The equation

$$N(t) = \frac{500}{1 + 49e^{-0.7t}}$$

models the number of people in a small town who have heard a rumor after t days.

1) What's the population of the small town?
2) How many people started the rumor?
3) To the nearest whole number, how many people will have heard the rumor after 3 days?
4) How many days will it take for 100 people to hear the rumor?

Solution.

1) When $t \to 00$, $e^{-0.7t} \to 0$ and
$$N(t) \to \frac{500}{1 + \underline{\qquad}} = \underline{\qquad}.$$
Therefore, the population of the small town is 500 people.

2) The number of people who started the rumor is given by:
$$N(0) = \frac{500}{1 + 49e^{0}} = \frac{500}{\underline{\qquad}} = \underline{\qquad}.$$

3) After 3 days:
$$N(3) = \frac{500}{1 + 49e^{-0.7(\underline{\qquad})}} = \frac{500}{1 + 49e^{-2.1}} \approx \underline{\qquad}.$$
Therefore, approximately 71 people will have heard the rumor after 3 days.

4) The time t it takes for 100 people to hear the rumor satisfies:

$$100 = \frac{500}{1 + 49e^{-0.7t}}.$$

Solve for t:

$$\frac{1}{5} = \frac{1}{1 + 49e^{-0.7t}}$$

$$1 + 49e^{-0.7t} = \_\_\_\_\_\_$$

$$e^{-0.7t} = \_\_\_\_\_\_$$

$$\_\_\_\_\_\_ = \ln\left(\frac{4}{49}\right)$$

$$t = \frac{\ln\left(\frac{49}{4}\right)}{0.7} \approx \_\_\_\_\_\_.$$

Therefore, it will take approximately 4 days for 100 people to hear the rumor.

Exercises

Exercise 3.5.1. A bacteria culture initially contains 3000 bacteria and doubles every half hour. Find the size of the bacteria population after 80 minutes.

Answer: Approximately 19049 bacteria.

Exercise 3.5.2. The half-life of tritium-3 is 12.25 years. How long would it take the sample to decay to 20% of its original amount? Round your answer to the nearest hundredth.

Answer: Approximately 28.44 years.

Exercise 3.5.3. A doctor prescribes 125 milligrams of a therapeutic drug that ***decays by*** about 30% each hour.

1) To the nearest hour, what is the half-life of the drug?
2) To the nearest hundredth hours, how long would it take the drug to ***decay to*** 30% of its original amount.

Answer: 1) Approximately 2 hours. 2) Approximately 3.38 hours.

Exercise 3.5.4. A cup of coffee at $185°\mathrm{F}$ is placed into a $60°\mathrm{F}$ room. One hour later, the temperature of coffee has dropped to $120°\mathrm{F}$. How long will it take for the temperature to drop to $80°\mathrm{F}$? Round your answer to the nearest minute.

Answer: Approximately 150 minutes.

Exercise 3.5.5. The population of a fish farm in t years is modeled by the equation

$$P(t) = \frac{1000}{1+9e^{-0.6t}}.$$

1) What is the initial population of fish?

2) What is the carrying capacity for the fish population?

3) To the nearest tenth, what is the doubling time for the fish population?

Answer: 1) The initial population is 100 fish. 2) The carrying capacity is 1000 fish. 3) Approximately 1.4 years.

Chapter 4 Trigonometric Functions

4.1 Review on Angles

Definition 4.1.1 (Angles and Measurements)

An **angle** is the union of two rays that share a common endpoint. This endpoint is the **vertex**, and the rays are the sides of the angle.

An angle can be formed by rotating a ray around its endpoint. The starting ray is the **initial side**, and the ending ray is the **terminal side**.

The **measure of an angle** is the amount of rotation from the initial side to the terminal side. A counterclockwise rotation produces a **positive angle**, while a clockwise rotation produces a **negative angle**.

One **degree** represents $\frac{1}{360}$ of a full counterclockwise rotation.

One **radian** is the measure of a central angle whose intercepted arc length equals the radius of the circle.

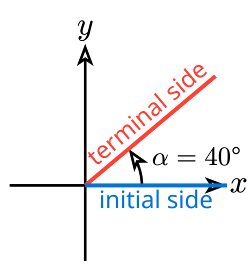
Conversion Factor: A half revolution, $180°$, is equivalent to π radians:

$$180° = \pi \text{ radians}, \quad \text{or} \quad 1 = \frac{180°}{\pi \text{ radians}}$$

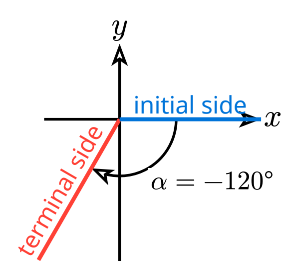
where $\pi \approx 3.14159$.

An angle is in **standard position** when its vertex is at the origin and its initial side lies along the positive x-axis.

A Positive Angle of $40°$ in Standard Position



A Negative Angle of $-120°$ in Standard Position



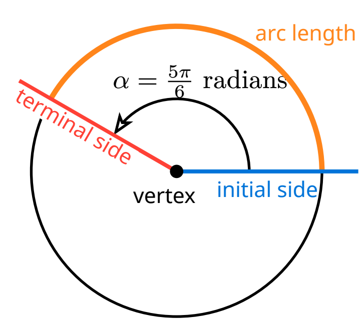
A **central angle** is an angle whose vertex is at the center of a circle.

An **arc** is any portion of a circle.

A **sector** is a region enclosed by two radii and the arc between them.

The length of an entire circle is its **circumference**. The length of an arc is called its **arc length**. The area of a sector is called its **sector area**.

A Central Angle of $\frac{5\pi}{6}$ radians $= 150°$



Example 4.1.1. Convert each radian measure to degrees and each degree measure to radians.

1) $\frac{\pi}{3}$ 2) 2 3) $36°$ 4) $150°$

Solution.

1) $\frac{\pi}{3} = \frac{\pi}{3} \cdot \frac{180°}{\pi} = \underline{\qquad\qquad}$

2) $2 = 2 \cdot \frac{180°}{\underline{\qquad}} = \underline{\qquad\qquad}$

3) $36° = 36° \cdot \frac{\pi}{\underline{\qquad}} = \underline{\qquad\qquad}$

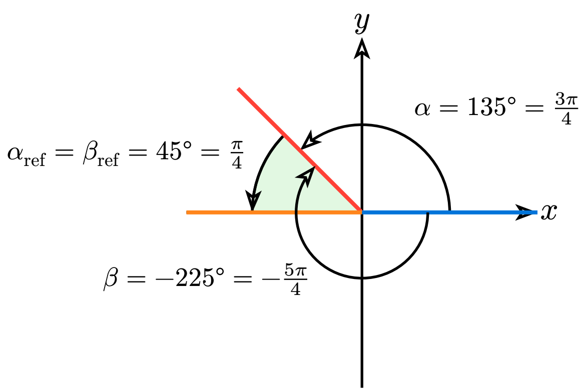
4) $150° = 150° \cdot \underline{\qquad\qquad} = \underline{\qquad\qquad}$

Definition 4.1.2 (Coterminal and Reference Angles)

Coterminal angles are angles in standard position that share the same terminal side.

The **reference angle** of an angle in the standard position is the acute angle (between 0 and $\frac{\pi}{2}$, or $0°$ and $90°$) formed by the terminal side of the angle and the positive or negative side of the x-axis.

The following figure illustrates coterminal angles and reference angles, where α and β are coterminal angles, and α_{ref} and β_{ref} are their reference angles.



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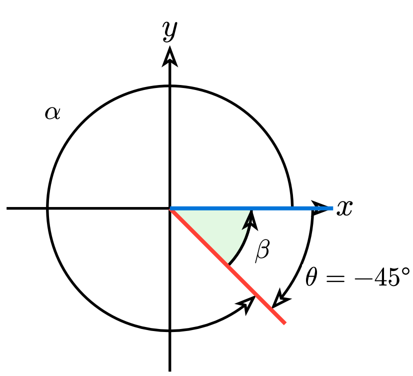
Example 4.1.2. Find a coterminal angle α such that $0° < \alpha < 360°$ and the reference angle β for the angle $\theta = -45°$.

Solution. To find the coterminal angle and the reference angle, it is better to draw a figure first.



From the figure, the coterminal angle is

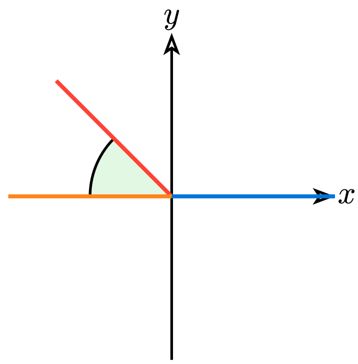
$$\alpha = 360° + \theta = 360° - 45° = \underline{\qquad\qquad}.$$

The reference angle is

$$\beta = -\alpha = \underline{\qquad\qquad}.$$

Example 4.1.3. Find a coterminal angle α such that $0 \le \alpha < 2\pi$ and the reference angle β for the angle $\theta = \frac{11\pi}{4}$.

Solution. To find the coterminal angle and the reference angle, it is better to draw a figure first.



Because $495 = \frac{11\pi}{4} > 2\pi$, we need to subtract 2π to find the coterminal angle α: From the figure, the coterminal angle is

$$\alpha = 495 - 2\pi = \frac{11\pi}{4} - 2\pi = \underline{\hspace{3cm}}.$$

From the figure, the reference angle is

$$\beta = \pi - \alpha = \pi - \underline{\hspace{2cm}} = \underline{\hspace{3cm}}.$$

Formulas for Arc Length and Sector Area

Let θ be the radian measure of a central angle in a circle of radius r.

- The **arc length** s of the angle is
$$s = r\theta.$$
- The **sector area** A enclosed by the angle and the arc is
$$A = \frac{1}{2}r^2\theta.$$

Example 4.1.4. Find the arc length of a central angle of 215 degrees in a circle of radius 10.

Solution. To find the arc length, we first convert the angle measure from degrees to radians:

$$\theta = 215° \cdot \frac{\pi}{180°} = \underline{\hspace{3cm}}.$$

Then, we use the arc length formula to find the arc length:

$$s = r\theta = 10 \cdot \underline{\hspace{3cm}} = \underline{\hspace{3cm}}.$$

Example 4.1.5. Find the sector area of a central angle of 150 degree in a circle of radius 12.

Solution. To find the sector area, we first convert the angle measure from degrees to radians:

$$\theta = 150° \cdot \frac{\pi}{180°} = \underline{\hspace{3cm}}.$$

Then, we use the sector area formula to find the sector area:

$$A = \frac{1}{2}r^2\theta = \frac{1}{2} \cdot 12^2 \cdot \underline{\hspace{3cm}} = \underline{\hspace{3cm}}.$$

Exercises

Find a coterminal angle α in degrees such that $0° \leq \alpha < 360°$ and the reference angle β in radians for the given angle.

1) $\theta = -120°$ 2) $\theta = 400°$ 3) $\theta = \frac{8\pi}{3}$ 4) $\theta = -\frac{5\pi}{4}$

Answer: 1) $\alpha = \frac{4\pi}{3}, \beta = \frac{\pi}{3}$ 2) $\alpha = \frac{2\pi}{9}, \beta = \frac{2\pi}{9}$ 3) $\alpha = \frac{2\pi}{3}, \beta = \frac{\pi}{3}$ 4) $\alpha = \frac{3\pi}{4}, \beta = \frac{\pi}{4}$

Exercise 4.1.1. A central angle in a circle of radius 3 is measure $120°$. Find the arc length on the circle and the sector area in the circle that are determined by the angle.

Answer: Arc length: 2π. Sector area: 3π.

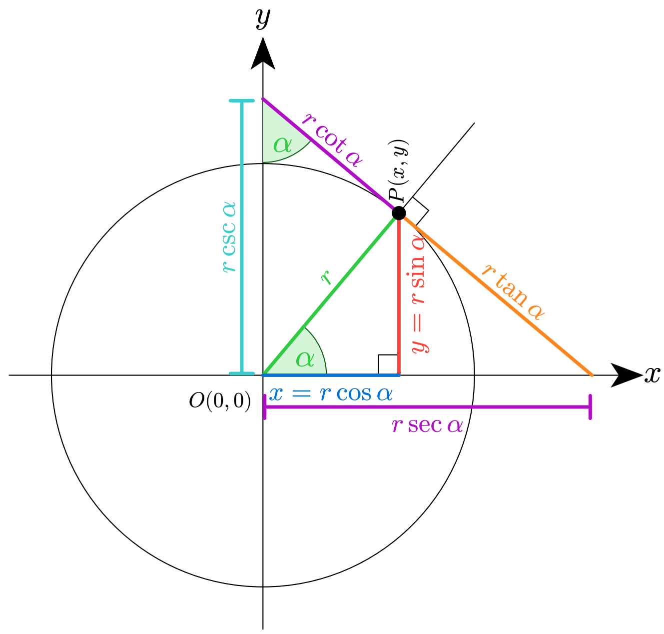
4.2 Trigonometric Functions of Arbitrary Angles

Definition 4.2.1 (Trigonometric Functions of Arbitrary Angles)

Let θ be a angle in the standard position and $P(x, y)$ is a point on the terminal side. Denote by r the distance between P and the origin O. Then $r = \sqrt{x^2 + y^2}$.

The trigonometric functions of the angle θ are defined as follows.

Sine	$\sin\theta = \dfrac{y}{r}$
Cosine	$\cos\theta = \dfrac{x}{r}$
Tangent	$\tan\theta = \dfrac{\sin\theta}{\cos\theta} = \dfrac{y}{x}$
Cosecant	$\csc\theta = \dfrac{1}{\sin\theta} = \dfrac{r}{y}$
Secant	$\sec\theta = \dfrac{1}{\cos\theta} = \dfrac{r}{x}$
Cotangent	$\cot\theta = \dfrac{1}{\tan\theta} = \dfrac{x}{y}$



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Note on Undefined Trigonometric Functions

In the above definitions, some trigonometric functions may be undefined for certain angles. For example, if the terminal side of angle θ lies along the y-axis, then $x = 0$ and both $\tan\theta$ and $\sec\theta$ are undefined. Similarly, if the terminal side of angle θ lies along the x-axis, then $y = 0$ and both $\cot\theta$ and $\csc\theta$ are undefined.

Polar Coordinate System

The definitions of the trigonometric functions above induce a coordinate system called the **polar coordinate system**. In this system, a point P in the plane is represented by an ordered pair (r, θ), where r is the distance from the origin to the point P, and θ is the angle formed by the positive x-axis and the line segment from the origin to the point P. The value of r can be positive, or zero. For a point P with polar coordinates (r, θ), its rectangular coordinates (x, y) can be found using the formulas

$$x = r\cos\theta$$
$$y = r\sin\theta.$$

In particular, if P is a point on the **unit circle**, that is the circle centered at the origin with the radius $r = 1$, then $\sin\theta = y$ and $\cos\theta = x$.

Example 4.2.1. Find the EXACT VALUES of all six trigonometric functions of the central angle θ whose terminal side passes through the point $\left(-\frac{1}{2}, -\frac{\sqrt{3}}{2}\right)$ on the unit circle.

Solution. Since the point is on the unit circle, we have $r = 1$. Using the definitions of the trigonometric functions, we have

$$\sin\theta = -\frac{\sqrt{3}}{2} \qquad \cos\theta = -\frac{1}{2} \qquad \tan\theta = \frac{\sin\theta}{\cos\theta} = \frac{-\frac{\sqrt{3}}{2}}{-\frac{1}{2}} = \underline{\hspace{2cm}}$$

$$\csc\theta = \frac{1}{\sin\theta} = \underline{\hspace{2cm}} \qquad \sec\theta = \frac{1}{\cos\theta} = \underline{\hspace{2cm}} \qquad \cot\theta = \frac{1}{\tan\theta} = \underline{\hspace{2cm}}$$

Example 4.2.2. Find the EXACT VALUES of all six trigonometric functions of the angle θ in the standard position whose terminal side passes through the point $(-3, -4)$.

Solution. By Pythagorean identity, the distance from the point to the origin is

$$r = \sqrt{\underline{\hspace{2cm}}^2 + \underline{\hspace{2cm}}^2} = \sqrt{25} = \underline{\hspace{2cm}}.$$

Using the definitions of the trigonometric functions, we have

$$\sin\theta = \underline{\hspace{2cm}} \qquad \cos\theta = \underline{\hspace{2cm}} \qquad \tan\theta = \underline{\hspace{2cm}}$$

$$\csc\theta = \underline{\hspace{2cm}} \qquad \sec\theta = \underline{\hspace{2cm}} \qquad \cot\theta = \underline{\hspace{2cm}}$$

Example 4.2.3. The terminal side of an angle θ in the standard position is in the third quadrant and the y-coordinate of the intersection of the terminal side with the unit circle is $-\frac{\sqrt{2}}{2}$. Find the x-coordinate of the point of intersection and then find the EXACT VALUES of all six trigonometric functions of the angle θ.

Solution. Since the point is on the unit circle, we have $r = 1$. Using the Pythagorean Theorem, we have

$$x^2 + \left(-\frac{\sqrt{2}}{2}\right)^2 = 1^2,$$

which gives

$$x^2 = 1 - \underline{\hspace{2cm}} = \underline{\hspace{2cm}}.$$

Thus,

$$x = \pm\sqrt{\frac{1}{2}} = \pm\frac{\underline{\hspace{2cm}}}{2}.$$

Since the terminal side of the angle is in the third quadrant, where both x- and y-coordinates are negative, we have

$$x = \underline{\hspace{2cm}}.$$

Therefore, the point of intersection is $\left(-\frac{1}{2}, -\frac{\sqrt{2}}{2}\right)$. Using the definitions of the trigonometric functions, we have

$$\sin\theta = \underline{\hspace{2cm}} \qquad \cos\theta = \underline{\hspace{2cm}} \qquad \tan\theta = \underline{\hspace{2cm}}$$

$$\csc\theta = \underline{\hspace{2cm}} \qquad \sec\theta = \underline{\hspace{2cm}} \qquad \cot\theta = \underline{\hspace{2cm}}$$

Example 4.2.4. Simplify the expression using the definition of trigonometric functions.

1) $\frac{\sec\theta}{\tan\theta}$.

2) $\tan t \csc t$

Solution. Let $P(x, y)$ be a point on the terminal side of angle θ or t and r is the distance from P to the origin. By definition, we have

1) $$\frac{\sec\theta}{\tan\theta} = \frac{\underline{\qquad}}{\underline{\qquad}} = \frac{r}{y} = \underline{\qquad}.$$

2) $$\tan t \csc t = \underline{\qquad} \cdot \underline{\qquad} = \frac{r}{x} = \underline{\qquad}.$$

Pythagorean Identities of Trigonometric Functions

For any angle θ, the following identities hold:

$$\sin^2\theta + \cos^2\theta = 1$$
$$1 + \tan^2\theta = \sec^2\theta$$
$$1 + \cot^2\theta = \csc^2\theta$$

Example 4.2.5. Given that $\sec t = -\frac{17}{8}$ and $0 < t < \pi$, find the EXACT VALUES of the other five trigonometric functions.

Solution. (Using Pythagorean Identities). Since $\sec t = -\frac{17}{8}$, we have $\cos t = -\frac{8}{17}$. Using the Pythagorean identity $\sin^2 t + \cos^2 t = 1$, we have

$$\sin^2 t + \left(-\frac{8}{17}\right)^2 = 1,$$

which gives

$$\sin^2 t = 1 - \underline{\qquad} = \underline{\qquad}.$$

Thus,

$$\sin t = \pm\sqrt{\frac{225}{289}} = \pm\frac{\underline{\qquad}}{17}.$$

Since $0 < t < \pi$, where the sine function is positive, we have

$$\sin t = \underline{\qquad}.$$

Therefore, we have

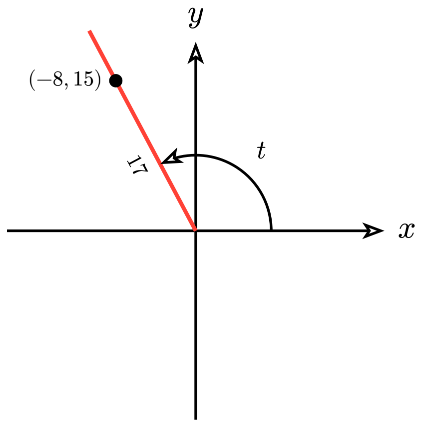
$$\tan t = \frac{\sin t}{\cos t} = \frac{\underline{\qquad}}{\underline{\qquad}} = \underline{\qquad},$$

$$\csc t = \frac{1}{\sin t} = \underline{\qquad},$$

$$\cot t = \frac{1}{\tan t} = \underline{\qquad}.$$

Solution. (Using the Definition). Since $\sec t = -\frac{17}{8}$, we have $\frac{r}{x} = -\frac{17}{8}$. Consider the point $(-8, y)$ such that $r = 17$. Since $0 < t < \pi$, we have $y > 0$. Using the Pythagorean Theorem, we have

$$y = \sqrt{r^2 - x^2} = \sqrt{(\underline{\qquad})^2 - (\underline{\qquad})^2} = \underline{\qquad}.$$



Therefore, we have

$$\sin t = \frac{y}{r} = \underline{\qquad},$$

$$\cos t = \frac{x}{r} = \underline{\qquad},$$

$$\tan t = \frac{\sin t}{\cos t} = \frac{\underline{\qquad}}{\underline{\qquad}} = \underline{\qquad},$$

$$\csc t = \frac{1}{\sin t} = \underline{\qquad},$$

$$\cot t = \frac{1}{\tan t} = \underline{\qquad}.$$

ⓘ Evaluate Trigonometric Functions using Reference Angles

To evaluate the trigonometric functions of an angle θ in standard position, we can use the reference angle θ_{ref} and the signs of the trigonometric functions in the quadrant where the terminal side of the angle θ lies.

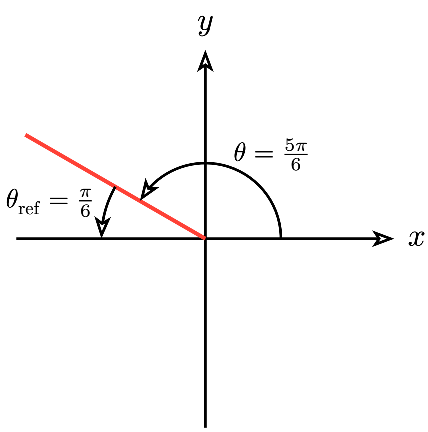
For example, if the terminal side of angle θ ***is in the second quadrant***, then

$$\sin\theta = \sin\theta_{\text{ref}} = \sin(\pi - \theta), \qquad \cos\theta = -\cos\theta_{\text{ref}} = -\cos(\pi - \theta)$$

Example 4.2.6. Use the reference angle to find the EXACT VALUES of all six trigonometric functions of $\frac{5\pi}{6}$

Solution. Let θ be the angle measured $\frac{5\pi}{6}$. Then the terminal side of the angle θ is in the second quadrant, the reference angle is

$$\theta_{\text{ref}} = \underline{\qquad} - \theta = \pi - \frac{5\pi}{6} = \underline{\qquad}.$$



Therefore,

$$\sin\theta = \sin\theta_{\text{ref}} = \underline{\qquad},$$

$$\cos\theta = -\cos\theta_{\text{ref}} = \underline{\qquad},$$

$$\tan\theta = \frac{\sin\theta}{\cos\theta} = \frac{\underline{\qquad}}{\underline{\qquad}} = \underline{\qquad},$$

$$\csc\theta = \frac{1}{\sin\theta} = \frac{1}{\underline{\qquad}} = \underline{\qquad},$$

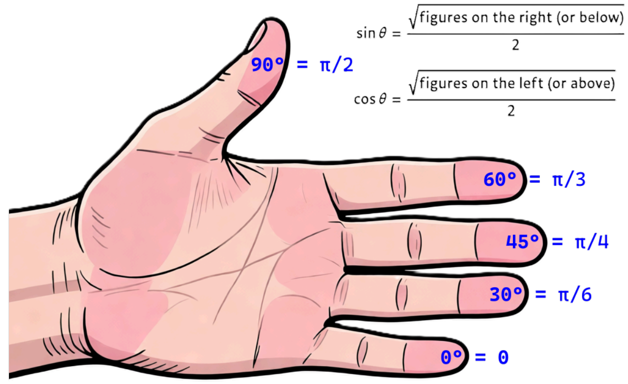
$$\sec\theta = \frac{1}{\cos\theta} = \frac{1}{\underline{\qquad}} = \underline{\qquad},$$

$$\cot\theta = \frac{1}{\tan\theta} = \frac{1}{\underline{\qquad}} = \underline{\qquad}.$$

Left-Hand Trick for Trigonometric Functions of Special Angles

To remember the sine and cosine values of the special angles 0, $\frac{\pi}{6}=30°$, $\frac{\pi}{4}=45°$, $\frac{\pi}{3}=60°$, and $\frac{\pi}{2}=90°$, use your hand as follows:

- Hold your left hand in front of you with your palm facing you and fingers spread apart.
- Label your thumb as $\frac{\pi}{2}$, index finger as $\frac{\pi}{3}$, middle finger as $\frac{\pi}{4}$, ring finger as $\frac{\pi}{6}$, and pinky as 0. The angles increase counter-clockwise from the pinky to the thumb.
- For a given angle θ, bend the finger corresponding to that angle.



- To find ***sine*** of an angle θ, count the ***fingers to the right of (or below) the bent finger***:

$$\sin\theta = \frac{\sqrt{\text{fingers on the right (or below)}}}{2}.$$

- To find ***cosine*** of an angle θ, count the ***fingers to the left of (or above) the bent finger***:

$$\cos\theta = \frac{\sqrt{\text{fingers on the left (or above)}}}{2}.$$

Example 4.2.7. Use the hand trick to find the EXACT VALUES of $\sin\frac{\pi}{3}$ and $\cos\frac{\pi}{3}$.

Solution. Bend the index finger to represent the angle $\frac{\pi}{3}$.

There are ______ fingers to the right of the bent finger. Thus,

$$\sin\frac{\pi}{3} = \frac{\sqrt{\underline{\qquad}}}{2}.$$

There are ______ fingers to the left of the bent finger. Thus,

$$\cos\frac{\pi}{3} = \frac{\sqrt{\underline{\qquad}}}{2}.$$

Symmetries of Trigonometric Functions

The trigonometric functions have the following symmetries.

- Cosine and secant are even functions:

$$\cos(-\theta)=\cos\theta \qquad \sec(-\theta)=\sec\theta.$$

- Sine, tangent, cosecant, and cotangent are odd functions:

$$\sin(-\theta)=-\sin\theta \qquad \tan(-\theta)=-\tan\theta \qquad \csc(-\theta)=-\csc\theta \qquad \cot(-\theta)=-\cot\theta$$

Example 4.2.8. Find all six trigonometric functions of the angle $-120°$ using the symmetries of trigonometric functions.

Solution. By symmetry and the reference angle method, we have

$$\sin(-120°) = -\sin 120° = -(\sin 60°) = \underline{\qquad},$$
$$\cos(-120°) = \cos 120° = -\cos 60° = \underline{\qquad},$$
$$\tan(-120°) = \underline{\qquad\qquad\qquad},$$
$$\csc(-120°) = \underline{\qquad\qquad\qquad},$$
$$\sec(-120°) = \underline{\qquad\qquad\qquad},$$
$$\cot(-120°) = \underline{\qquad\qquad\qquad}.$$

Definition 4.2.2 (Periodic Function)

A function f is called a **periodic function** if there is number p such that $f(x+p) = f(x)$ for all x. The smalled positive number p such that $f(x+p) = f(x)$ for all x is called the **period** of the function f.

Periods of Trigonometric Functions

The period of the cosine, sine, secant, and cosecant functions is 2π.

The period of the tangent and cotangent functions is π.

Example 4.2.9. Find the EXACT Values of the six trigonometric functions of the angle $\theta = \frac{7\pi}{3}$ using the periodicity of trigonometric functions.

Solution. Since $\frac{7\pi}{3} = 2\pi + \frac{\pi}{3}$, by periodicity, we have

$$\sin\frac{7\pi}{3} = \sin\left(2\pi + \frac{\pi}{3}\right) = \sin\frac{\pi}{3} = \underline{\qquad},$$
$$\cos\frac{7\pi}{3} = \cos\left(2\pi + \frac{\pi}{3}\right) = \cos\frac{\pi}{3} = \underline{\qquad},$$
$$\tan\frac{7\pi}{3} = \tan\left(2\pi + \frac{\pi}{3}\right) = \tan\frac{\pi}{3} = \underline{\qquad},$$
$$\csc\frac{7\pi}{3} = \csc\left(2\pi + \frac{\pi}{3}\right) = \csc\frac{\pi}{3} = \underline{\qquad},$$
$$\sec\frac{7\pi}{3} = \sec\left(2\pi + \frac{\pi}{3}\right) = \sec\frac{\pi}{3} = \underline{\qquad},$$
$$\cot\frac{7\pi}{3} = \cot\left(2\pi + \frac{\pi}{3}\right) = \cot\frac{\pi}{3} = \underline{\qquad}.$$

Exercises

Exercise 4.2.1. Find the coordinates of the point on the unit circle and the terminal side of the given angle. Show your answer in exact form.

1) $\theta = 225°$ 2) $\theta = \frac{3\pi}{4}$ 3) $\theta = \frac{11\pi}{6}$

Answer: 1) $\left(-\frac{\sqrt{2}}{2}, -\frac{\sqrt{2}}{2}\right)$ 2) $\left(-\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right)$ 3) $\left(\frac{\sqrt{3}}{2}, -\frac{1}{2}\right)$

Exercise 4.2.2. Find all six trigonometric functions of the angle in the standard position whose terminal side passing through the given point. Show your answer in exact form.

1) $\left(\frac{\sqrt{3}}{2}, -\frac{1}{2}\right)$ 2) $(-1, 2)$

Answer:
1) $\sin\theta = -\frac{1}{2}, \cos\theta = \frac{\sqrt{3}}{2}, \tan\theta = -\frac{\sqrt{3}}{3}, \csc\theta = -2, \sec\theta = \frac{2}{\sqrt{3}} = \frac{2\sqrt{3}}{3}, \cot\theta = -\sqrt{3}$
2) $\sin\theta = \frac{2}{\sqrt{5}} = \frac{2\sqrt{5}}{5}, \cos\theta = -\frac{1}{\sqrt{5}} = -\frac{\sqrt{5}}{5}, \tan\theta = -2, \csc\theta = \frac{\sqrt{5}}{2}, \sec\theta = -\sqrt{5}, \cot\theta = -\frac{1}{2}$

Exercise 4.2.3. Given that $\tan\theta = -2$ and $-\frac{\pi}{2} < \theta < \frac{\pi}{0}$, find the EXACT VALUES of the other five trigonometric functions. Show your answer in exact form.

Answer: $\sin\theta = -\frac{2}{\sqrt{5}} = -\frac{2\sqrt{5}}{5}, \cos\theta = \frac{1}{\sqrt{5}} = \frac{\sqrt{5}}{5}, \csc\theta = -\frac{\sqrt{5}}{2}, \sec\theta = \sqrt{5}, \cot\theta = -\frac{1}{2}$

Exercise 4.2.4. Simplify the expression.

1) $\frac{\cot\theta}{\csc\theta}$

2) $\sec\theta\tan\theta\cos^2\theta$

Answer: 1) $\cos\theta$ 2) $\sin\theta$

Exercise 4.2.5. Find all six trigonometric functions of each angle. Show your answer in exact form.

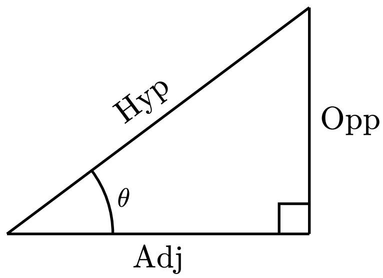
1) $A = \frac{4\pi}{3}$ 2) $B = -\frac{5\pi}{6}$ 3) $C = 750°$

Answer:
1) $\sin A = -\frac{\sqrt{3}}{2}, \cos A = -\frac{1}{2}, \tan A = \sqrt{3}, \csc A = -\frac{2}{\sqrt{3}} = \frac{2\sqrt{3}}{3}, \sec A = -2, \cot A = \frac{1}{\sqrt{3}} = \frac{\sqrt{3}}{3}$
2) $\sin B = -\frac{1}{2}, \cos B = -\frac{\sqrt{3}}{2}, \tan B = \frac{\sqrt{3}}{3}, \csc B = -2, \sec B = -\frac{2}{\sqrt{3}} = -\frac{2\sqrt{3}}{3}, \cot B = \sqrt{3}$
3) $\sin C = \frac{1}{2}, \cos C = \frac{\sqrt{3}}{2}, \tan C = \frac{\sqrt{3}}{3}, \csc C = 2, \sec C = \frac{2}{\sqrt{3}} = \frac{2\sqrt{3}}{3}, \cot C = \sqrt{3}$

4.3 Right Triangle Trigonometry

Trigonometric Functions Associated to a Right Triangle

Given a right triangle with an acute angle θ, the six **trigonometric functions** and lengths of the sides are related as follows:

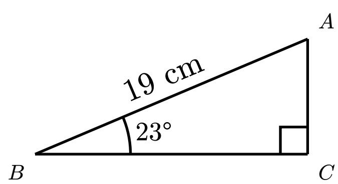


$$\sin\theta = \frac{\text{Opp}}{\text{Hyp}} \qquad \cos\theta = \frac{\text{Adj}}{\text{Hyp}} \qquad \tan\theta = \frac{\text{Opp}}{\text{Adj}}$$

$$\csc\theta = \frac{\text{Hyp}}{\text{Opp}} \qquad \sec\theta = \frac{\text{Hyp}}{\text{Adj}} \qquad \cot\theta = \frac{\text{Adj}}{\text{Opp}}$$

Example 4.3.1. In triangle $\triangle ABC$, if $\angle C = 90°$, $AB = 19$ cm and $\angle B = 23°$, determine the length of AC and the length of BC to the nearest tenth of a centimeter.

Solution.



The length AC and the trigonometric functions of $\angle B$ are related as follows:,

$$\sin\angle B = \frac{AC}{AB},$$

which gives

$$AC = (AB)\cdot\sin\angle B = 19\cdot\sin 23° = \underline{\qquad\qquad} \text{ cm}.$$

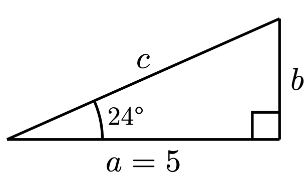
Also,

$$\cos\angle B = \frac{BC}{AB},$$

which gives

$$BC = (AB)\cdot\cos\angle B = 19\cdot\cos 23° = \underline{\qquad\qquad} \text{ cm}.$$

Example 4.3.2. Find sides b and c in the following right triangle.



Solution. Let θ be the angle opposite to side b. Then, we have $\theta = 24°$. The sides a, b and c in the right triangle are related to the trigonometric functions of angle θ as follows:

$$\tan\theta = \frac{b}{a}, \quad \text{and} \quad \sec\theta = \frac{c}{a}.$$

Thus, we have

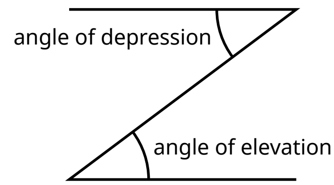
$$b = a\cdot\tan\theta = 5\cdot\tan 24° = \underline{\qquad\qquad},$$

$$c = a\cdot\sec\theta = 5\cdot\sec 24° = \underline{\qquad\qquad}.$$

Angle of elevation or depression

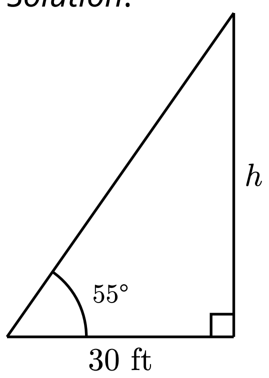
The **angle of elevation** is the angle formed by the horizontal line and the line of sight to an object above the horizontal line.

The **angle of depression** is the angle formed by the horizontal line and the line of sight to an object below the horizontal line.



Example 4.3.3. The angle of elevation to the top of a tall tree is $55°$ when measured at a point 30 feet from the base. Assume the ground is flat. How tall is the tree?

Solution.



Let h be the height of the tree. Then, we have

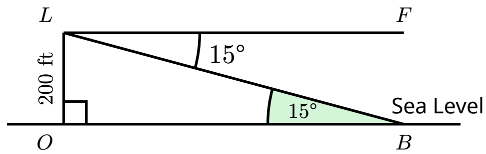
$$\tan 55° = \frac{\underline{\qquad}}{30},$$

which gives

$$h = 30 \cdot \tan 55° = \underline{\qquad\qquad} \text{ feet.}$$

Example 4.3.4. A lighthouse stands 200 feet above sea level. From the top of the lighthouse, a boat is observed at an angle of depression of $15°$. Assuming the sea surface is flat and horizontal and the lighthouse is perpendicular to the sea surface, how far is the boat from the top of the lighthouse?

Solution.



Let L be the top of the lighthouse, B be the position of the boat, and O be the intersection of the perpendicular lines through L and B, as shown in the figure on the left. Then

$$LO = 200 \quad \text{feet}, \quad \text{and} \quad \angle BLF = 15°.$$

Because $\angle OBL$ and $\angle BLF$ are alternate interior angles, we have $\angle OBL = 15°$. Therefore, the distance LB from the top of the lighthouse to the boat and the trigonometric functions of angle $\angle OLB$ are related as follows:

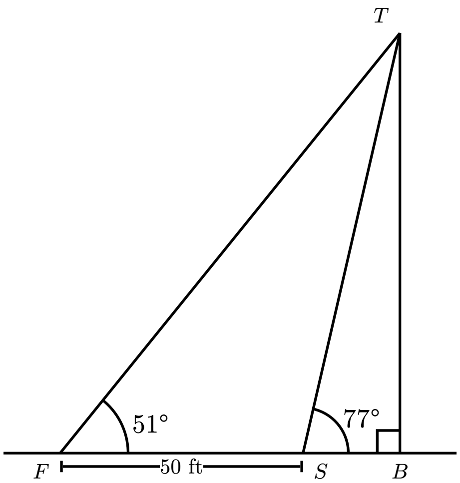
$$\sin 15° = \frac{\underline{\qquad}}{LB}.$$

Solving for LB, we get the distance from the top to the lighthouse to the boat is

$$LB = \frac{200}{\underline{\qquad}} = \underline{\qquad\qquad} \text{ feet.}$$

Example 4.3.5. To estimate the height of a building, two measurements are taken. The first measurement shows an angle of elevation to the top of the building as $51°$. The second measurement, taken 50 feet closer to the base of the building, yields an angle of elevation of $77°$. From the measurements, estimate the height of the building. **Round to the nearest foot.**

Solution.



Let T be the top of the building, B be the base of the building, and F and S be the first and the second observation points, as shown in the figure on the left. From the definition the tangent function, we have the following equations:

$$\tan 51° = \frac{TB}{FB}, \quad \tan 77° = \frac{TB}{SB}, \text{ and } \quad FB - SB = 50 \text{ feet}.$$

Solving for FB and SB in terms of TB and plugging them in to the third equation induces an equation in one variable TB as follows:

$$\frac{TB}{\underline{\qquad\qquad}} - \frac{TB}{\underline{\qquad\qquad}} = 50 \text{ feet}.$$

Solving for TB, we get the height of the building is

$$TB = \frac{50}{\frac{1}{\tan 51°} - \frac{1}{\tan 77°}} = \underline{\qquad\qquad} \text{ feet}.$$

Cofunction Identities

Given an angle θ measured in radians, we have the following cofunction identities.

$$\sin\left(\frac{\pi}{2} - \theta\right) = \cos\theta \qquad \csc\left(\frac{\pi}{2} - \theta\right) = \sec\theta$$

$$\cos\left(\frac{\pi}{2} - \theta\right) = \sin\theta \qquad \sec\left(\frac{\pi}{2} - \theta\right) = \csc\theta$$

$$\tan\left(\frac{\pi}{2} - \theta\right) = \cot\theta \qquad \cot\left(\frac{\pi}{2} - \theta\right) = \tan\theta$$

Reasoning: If θ is an acute angle in a right triangle, then $\frac{\pi}{2} - \theta$ is the other acute angle in the triangle. Therefore, these identities can be interpreted using right triangle trigonometry.

In general, this identities can be obtained using the reference angle method.

Example 4.3.6. If $\sin t = \frac{5}{13}$ and $0 < t < \frac{\pi}{2}$, find $\tan(\frac{\pi}{2} - t)$.

Solution. By the cofunction identity, we have

$$\tan\left(\frac{\pi}{2} - t\right) = \cot t = \frac{\cos t}{\sin t}.$$

Because $0 < t < \frac{\pi}{2}$, by the Pythagorean identity,

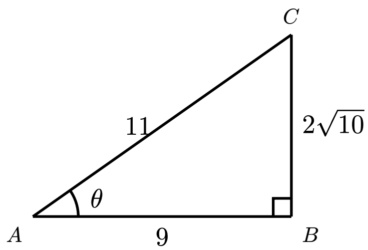
$$\cos t = \sqrt{1 - \sin^2 t} = \sqrt{1 - \left(\frac{5}{13}\right)^2} = \underline{\qquad\qquad}.$$

Thus, we have

$$\tan\left(\frac{\pi}{2} - t\right) = \frac{\cos t}{\sin t} = \frac{\underline{\qquad\qquad}}{\underline{\qquad\qquad}} = \frac{5}{\underline{\qquad\qquad}}.$$

Exercises

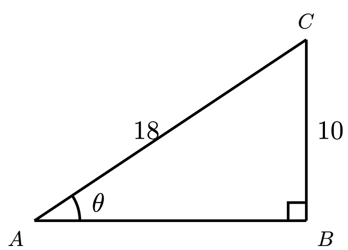
Exercise 4.3.1. Find all trigonometric functions of the angle θ in the right triangle given below.



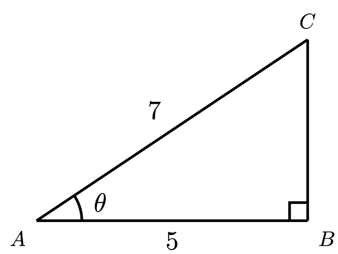
Answer: $\sin\theta = \frac{2\sqrt{10}}{11}$, $\cos\theta = \frac{9}{11}$, $\tan\theta = \frac{2\sqrt{10}}{9}$, $\csc\theta = \frac{11}{2\sqrt{10}} = \frac{11\sqrt{10}}{20}$, $\sec\theta = \frac{11}{9}$, $\cot\theta = \frac{9}{2\sqrt{10}} = \frac{9\sqrt{10}}{20}$.

Exercise 4.3.2. Find $\sin\theta$, $\cos\theta$ and $\tan\theta$ of the angle θ given in the figure.

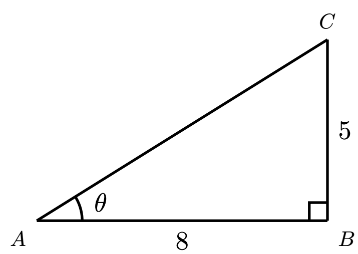
1)



2)

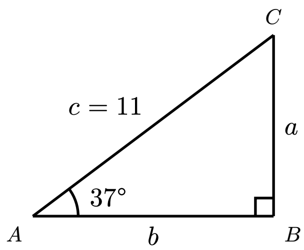


3)



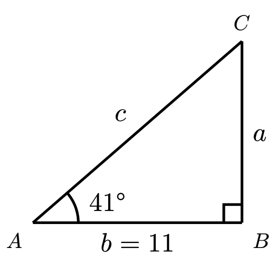
Answer:
1) $\sin\theta = \frac{5}{9}, \cos\theta = \frac{2\sqrt{14}}{9}, \tan\theta = \frac{5}{2\sqrt{14}}$,
2) $\sin\theta = \frac{2\sqrt{6}}{7}, \cos\theta = \frac{5}{7}, \tan\theta = \frac{2\sqrt{6}}{5}$,
3) $\sin\theta = \frac{5\sqrt{89}}{89}, \cos\theta = \frac{8\sqrt{89}}{89}, \tan\theta = \frac{5}{8}$.

Exercise 4.3.3. Find sides a and b in the following right triangle (round to the nearest thousandth).



Answer: $a = 6.62$, $b = 8.785$.

Exercise 4.3.4. Find sides a and c in the following right triangle (round to the nearest thousandth).



Answer: $a = 9.562$, $c = 14.575$.

Exercise 4.3.5. In triangle $\triangle ABC$, if $\angle C = 90°$, $AC = 52\,\text{cm}$ and $\angle B = 37°$, determine the length of AB and the length of BC to the nearest tenth of a centimeter.

Answer: $AB = 86.4$ cm, $BC = 69$ cm.

Exercise 4.3.6. A hot air balloon hovers above the ground at a height of 1000 feet. A person on the ground sees the balloon at an angle of elevation of $27°$. What is the distance between the balloon and the person? (Round to the nearest foot.)

Answer: 2203 feet.

Exercise 4.3.7. A jet takes off at a $20°$ angle. The runway from takeoff is 800 meters long. What is the altitude of the airplane when it flies over the end of the runway? (Round to the nearest tenth of a meter)

Answer: 291.2 meters.

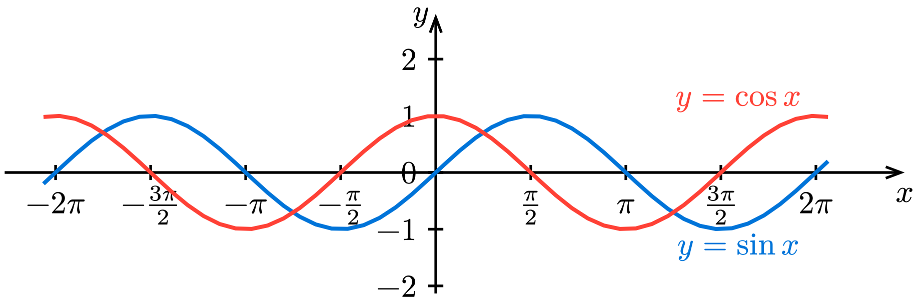
Exercise 4.3.8. If $\cos\alpha = \frac{12}{13}$, find the possible values of $\cos(\frac{\pi}{2} - \alpha)$.

Answer: $\cos(\frac{\pi}{2} - \alpha) = \pm\frac{5}{13}$

4.4 Graphs of Sine and Cosine

Properties of Standard Sine and Cosine Functions: $y = \sin x$ and $y = \cos x$

The functions $y = \sin x$ and $y = \cos x$ are called the **standard sine function** and the **standard cosine function**, respectively.



Common Properties of $y = \sin x$ and $y = \cos x$

Both functions are periodic with the **period** of 2π, **domain** $(-\infty, \infty)$, and **range** $[-1, 1]$.

In the following, k is any integer.

Properties of $y = \sin x$

- y-intercept: $(0, 0)$
- x-intercepts: $(k\pi, 0)$.
- Global (and local) maximum:
$$1 = \sin\left(2k\pi + \frac{\pi}{2}\right).$$
- Global (and local) minimum:
$$-1 = \sin\left(2k\pi - \frac{\pi}{2}\right).$$
- Symmetric with respect to the vertical line $x = k\pi + \frac{\pi}{2}$:
$$\sin\left(k\pi + \frac{\pi}{2} + x\right) = \sin\left(k\pi + \frac{\pi}{2} - x\right).$$
- Symmetric with respect to the intersection points $(k\pi, 0)$ with the midline:
$$\sin(k\pi + x) = -\sin(k\pi - x).$$
In particular, $y = \sin x$ is an ***odd*** function.

Properties of $y = \cos x$

- y-intercept: $(0, 1)$
- x-intercepts: $\left(k\pi + \frac{\pi}{2}, 0\right)$.
- Global (and local) maximum:
$$1 = \cos(2k\pi).$$
- Global (and local) minimum:
$$-1 = \cos(2k\pi + \pi).$$
- Symmetric with respect to the vertical line $x = k\pi$:
$$\cos(k\pi + x) = \cos(k\pi - x).$$
In particular, $y = \cos x$ is an ***even*** function.
- Symmetric with respect to the intersection points $(k\pi + \frac{\pi}{2}, 0)$ with the midline:
$$\cos\left(k\pi + \frac{\pi}{2} + x\right) = -\cos\left(k\pi + \frac{\pi}{2} - x\right).$$

Relationship between Sine and Cosine Functions

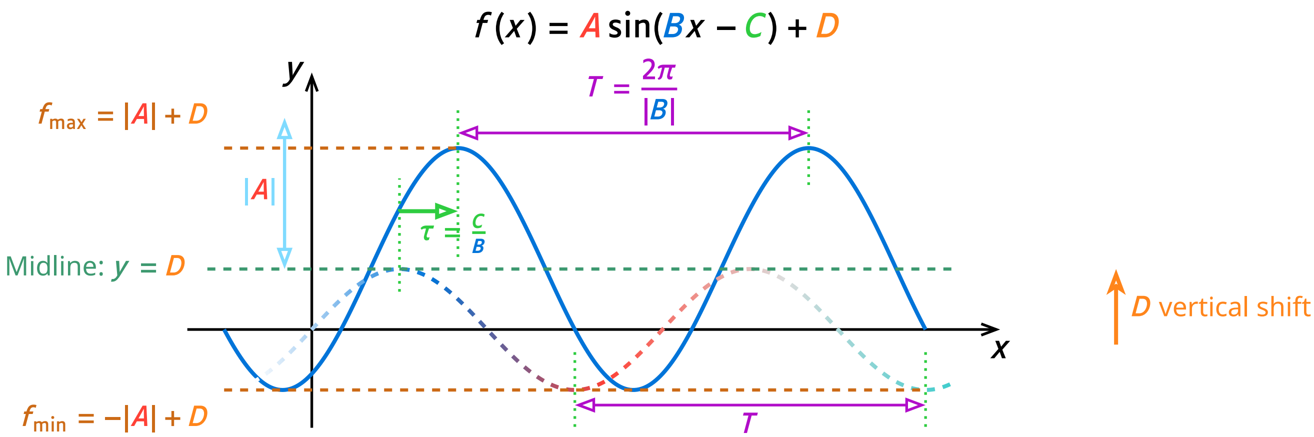
By the symmetry of the cosine function and the cofunction identity, we have
$$\cos x = \cos(-x) = \sin\left(\frac{\pi}{2} - (-x)\right) = \sin\left(x + \frac{\pi}{2}\right).$$
Graphically, $y = \cos x$ is a horizontal shift of $\frac{\pi}{2}$ units to the left of $y = \sin x$.

Definition 4.4.1 (Sinusoidal Functions and Their Characteristics)

A **sinusoidal function** is a function f defined by $f(x) = A\sin(Bx - C) + D$ or $f(x) = A\cos(Bx - C) + D$. They have the following characteristics:

- The **midline** is the horizontal axis of oscillation: $y = D$, where D is the average of the maximum and minimum values of f.
- The **amplitude** is the maximum vertical displacement from the midline, given by $|A|$, which also equals $\frac{|f_{\max} - f_{\min}|}{2}$, where $f_{\max}$ and $f_{\min}$ are the maximum and minimum of f.
- The **period** T is the smallest positive horizontal distance for one complete cycle, starting at the midline, passing through a maximum, then a minimum, and returning to the midline, such that $f(x+T) = f(x)$. It is calculated as $T = \frac{2\pi}{|B|}$.

  The period is also the distance between two consecutive maximums or two consecutive minimums or twice the distance of two consecutive midline crossings.
- The **phase shift**[4] τ is the horizontal shift (with in a period) relative to the standard sinusoidal function and given by $\tau = \frac{C}{B}$.



- General sinusoidal functions have **symmetries** similar to the standard ones. ♣

Example 4.4.1. Determine the midline, amplitude, period, and phase shift of the function $y = 3\sin(2x - \frac{\pi}{2}) + 1$.

Solution. Compare the function to the standard form $y = A\sin(Bx - C) + D$, we have

$$A = 3, \quad B = 2, \quad C = \frac{\pi}{2}, \quad \text{and} \quad D = 1.$$

Thus, we can determine the characteristics as follows:

Midline: $y =$ ______ Amplitude: ______

Period: $\frac{2\pi}{\_\_\_\_\_\_} =$ ______ Phase Shift: $\frac{\_\_\_\_\_\_}{2} =$ ______

[4]In general, a phase shift of f relative to g with the same period is a value τ such that $f(x) = Ag(x+\tau) + D$ for all x in the domain of f. Be careful, in Physics, the term "phase shift" may have a different meaning.

Example 4.4.2. Consider the function $y = -2\cos(\frac{\pi}{2}x + \pi) + 3$, determine the amplitude, period, phase shift, and midline. Then graph the function.

Solution. Compare the function to the standard form $y = A\cos(Bx - C) + D$, we have

$$A = -2, \quad B = \frac{\pi}{2}, \quad C = -\pi, \quad \text{and} \quad D = 3.$$

Thus, we can determine the characteristics as follows:

Midline: $y =$ __________ Amplitude: __________

Period: $\frac{2\pi}{\_\_\_\_\_\_} =$ __________ Phase Shift: $\frac{\_\_\_\_\_\_}{2} =$ __________

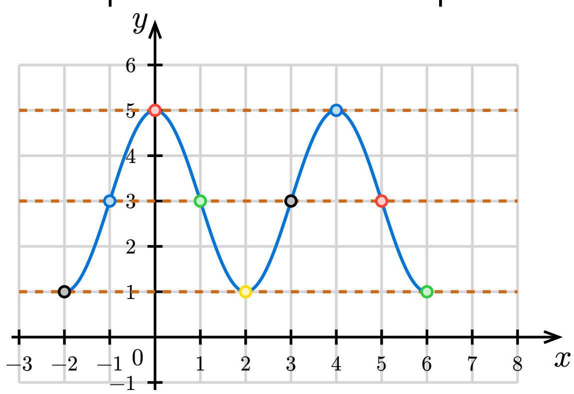
To sketch the graph, we first plot three dashed horizontal lines:

- the midline $y = 3$,
- the line $y = 3 + |A| =$ __________ passing through a minimum, and
- the line $y = 3 - |A| =$ __________ passing through a minimum.

Then, we plot key points starting from the phase shift $x = -2$ and moving to the right (or left) by the quarter period $\frac{T}{4} = \frac{\_\_\_\_\_\_}{4} =$ ______. The key points are:

- At $x = -2$, $y = 3 - |A| =$ ______ (minimum because $A < 0$)
- At $x = -2 + \frac{T}{4} =$ ______, $y = 3$ (midline because the function must be increasing)
- At $x = -2 + \frac{T}{2} =$ ______, $y = 3 + |A| =$ ______ (maximum)
- At $x = -2 + 3\frac{T}{4} =$ ______, $y = 3$ (midline because the function must be decreasing)
- At $x = -2 + T =$ ______, $y = 3 - |A| =$ ______ (minimum)

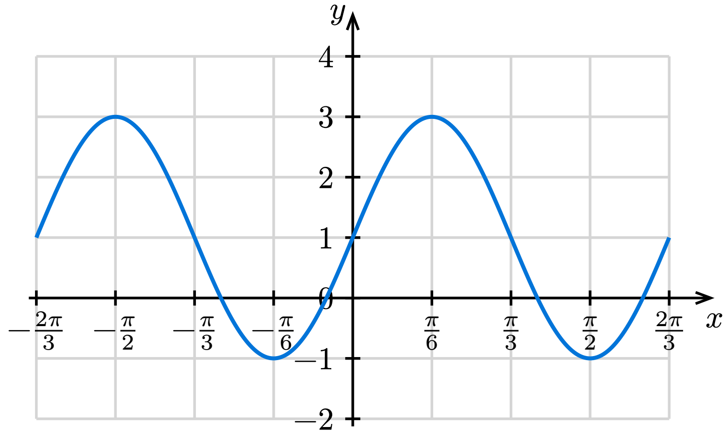
Finally, we connect the key points with a smooth curve to complete one period of the graph, and repeating the same pattern for additional periods.



Remark

The five key points can also be found by evaluating the function at $x = -2, -1, 0, 1, 2$, which are the values ***phase shift plus multiples of quarter period***.

Example 4.4.3. Find an equation of the sinusoidal function defined by the following graph.



Solution. We may assume the function is in the form of $f(x) = A\sin(Bx - C) + D$. Moreover, we may assume that $B > 0$, otherwise, we can replace B with $-B$ and C with $-C$ and move the negative sign to A using the identity $\sin(-x) = -\sin x$.

From the graph, we see the maximum value is $f_{\max} = 3$ and the minimum value is $f_{\min} = -1$. Thus, we have

$$D = \frac{f_{\max} + f_{\min}}{2} = 1, \quad |A| = \frac{f_{\max} - f_{\min}}{2} = 2.$$

Therefore, the midline is $y = 1$ and the amplitude is 2.

Starting to the point $(0, 1)$ on the midline and moving to the right, we see that one complete cycle ends at the point $\left(\frac{2\pi}{3}, 1\right)$. Thus, the period is $T =$ ______. Since B is assumed to be positive, the formula $T = \frac{2\pi}{B}$ gives

$$B = \frac{2\pi}{T} = \underline{\qquad}.$$

Since the y-intercept $(0, 1)$ is on the midline, comparing with that of $y = \sin x$, we have the phase shift is 0. Thus, we have

$$C = B \cdot \underline{\qquad} = 0.$$

Now we can write the equation of the sinusoidal function as

$$f(x) = A\sin(\underline{\qquad}\, x) + \underline{\qquad}.$$

Since the graph shows that the function is increasing at $x = 0$, we have $A > 0$. Therefore, the equation of the sinusoidal function is

$$f(x) = \underline{\qquad}\sin(\underline{\qquad}\, x - \underline{\qquad}) + \underline{\qquad}.$$

Remark

Assuming the function is in the form of $f(x) = A\cos(Bx - C) + D$, using a similar process, one can find that

$$f(x) = 2\cos\left(3x - \frac{\pi}{2}\right) + 1.$$

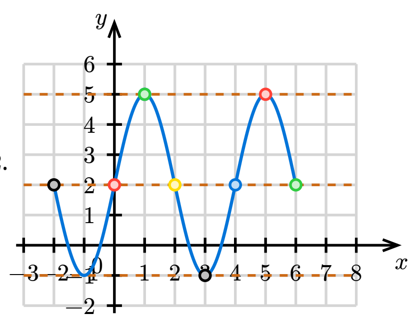
Exercises

Exercise 4.4.1. Determine the midline, amplitude, period, and phase shift of the function $y = 2\cos(2\pi x - \pi) - 1$.

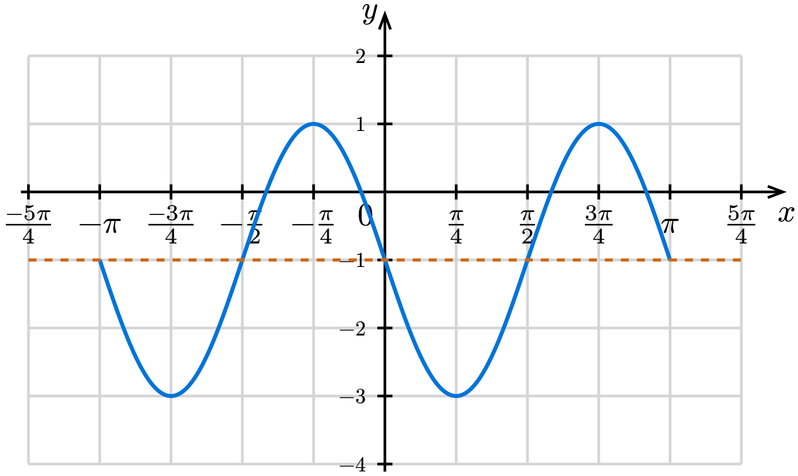
Answer: Midline: $y = -1$; Amplitude: 2; Period: $T = \frac{2\pi}{2\pi} = 1$; Phase Shift: $\frac{\pi}{2\pi} = \frac{1}{2}$.

Exercise 4.4.2. Given $y = -3\sin\left(\frac{\pi}{2}x - \pi\right) + 2$, determine the amplitude, period, phase shift, and midline. Then graph the function.

Answer: Midline: $y = 2$; Amplitude: 3; Period: 4; Phase Shift: -2.



Exercise 4.4.3. Find an equation of the sinusoidal function defined by the following graph.



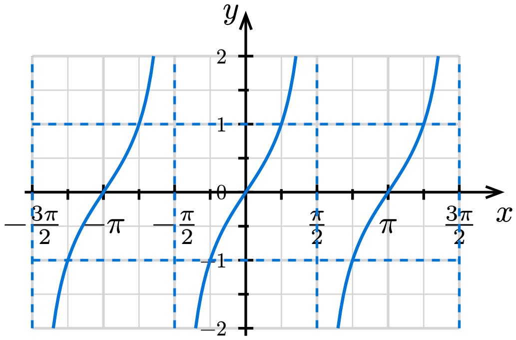
Answer: $f(x) = -2\sin(2x) - 1.$

4.5 Graph of Other Trigonometric Functions

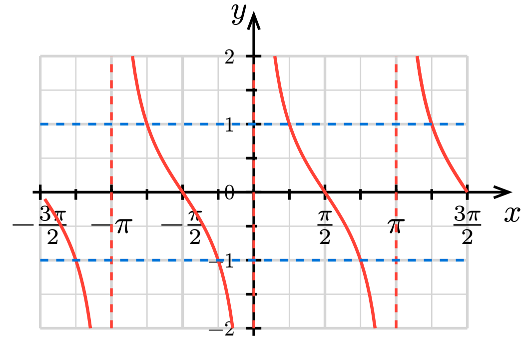
Properties of $y = A\tan(Bx - C) + D$ and $y = a\cot(Bx - C) + D$

Tangent and cotangent functions are periodic odd functions with vertical asymptotes.

Graph of the standard function $y = \tan x$



Graph of the standard function $y = \cot x$



Common Properties of $y = A\tan(Bx - C) + D$ and $y = a\cot(Bx - C) + D$

- Period: $\frac{\pi}{|B|}$, Range: $(-\infty, \infty)$, Phase shift: $\frac{C}{B}$, Midline: $y = D$.
- Special points:
$$\left(\frac{k\pi}{B} + \frac{\frac{\pi}{4} + C}{B}, A + D\right), \qquad \left(\frac{k\pi}{B} + \frac{-\frac{\pi}{4} + C}{B}, -A + D\right),$$
which are derived from $\tan\left(\frac{\pi}{4}\right) = \cot\left(\frac{\pi}{4}\right) = 1$, where k is any integer.
- Symmetric with respect to each intersection of the graph or the vertical asymptote with the midline because $\tan\left(\frac{k\pi}{2} + x\right) = -\tan\left(\frac{k\pi}{2} - x\right)$ and $\cot\left(\frac{k\pi}{2} + x\right) = -\cot\left(\frac{k\pi}{2} - x\right)$. In particular, **if $C = 0$ and $D = 0$**, that is, no horizontal and vertical shifts, then the functions are **odd** functions.

Properties of $y = A\tan(Bx - C) + D$

- Domain: all real numbers x such that $Bx - C \neq k\pi + \frac{\pi}{2}$.
- Vertical asymptotes:
$$x = \frac{k\pi}{B} + \frac{\pi + 2C}{2B}.$$
- Increasing (decreasing) within each interval (a, b) in its domain if $A > 0$ (if $A < 0$).
- Points on the midline:
$$\left(\frac{k\pi}{B} + \frac{C}{B}, D\right).$$

Properties of $y = a\cot(Bx - C) + D$

- Domain: all real numbers x such that $Bx - C \neq k\pi$.
- Vertical asymptotes:
$$x = \frac{k\pi}{B} + \frac{C}{B}.$$
- Decreasing (increasing) within each interval (a, b) in its domain if $A > 0$ (if $A < 0$).
- Points on the midline:
$$\left(\frac{k\pi}{B} + \frac{\pi + 2C}{B}, D\right).$$

The period is also the distance between two consecutive vertical asymptotes.

Example 4.5.1. Sketch a graph of one period of the function $y = \frac{1}{2}\tan(\frac{\pi}{2}x)$.

Solution. The graph can be sketched using properties of the function.

The period is

$$T = \frac{\pi}{\underline{\qquad}} = 2.$$

Since $y = \tan x$ has vertical asymptotes at $x = \pm\frac{\pi}{2}$, solving $\frac{\pi}{2}x = \pm\frac{\pi}{2}$ gives the vertical asymptotes of the function at

$$x = \pm \underline{\qquad}.$$

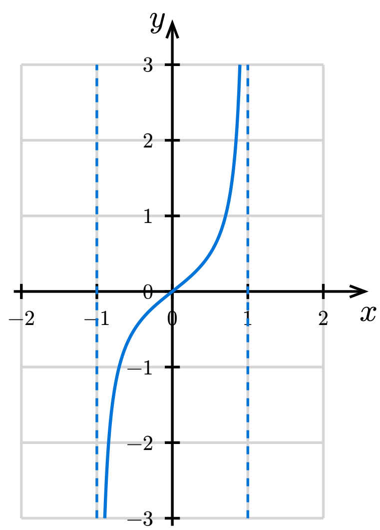
The midline is $y = 0$, and an x-intercept is at $(0, 0)$.

Since $\tan\left(\pm\frac{\pi}{4}\right) = \pm \underline{\qquad}$, solving $\frac{\pi}{2}x = \pm\frac{\pi}{4}$ gives the special points at

$$\left(\pm \underline{\qquad}, \pm\frac{1}{2}\right).$$

Because $A > 0$, the function is increasing in each interval of its domain.

Plotting the asymptotes, intercept, and special points, and connecting them with a smooth curve gives the graph of the function within one period.



Graph of Cotangent Functions

Recall that tangent and cotangent are cofunctions. Then

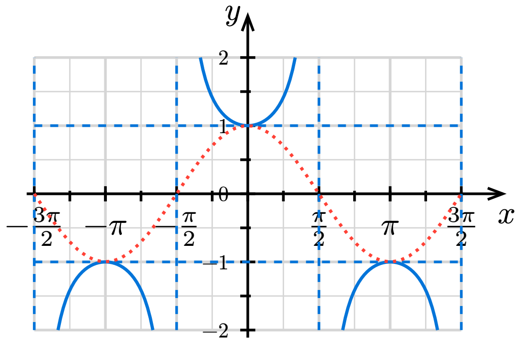
$$\tan(x) = \cot\left(\frac{\pi}{2} - x\right) = -\cot\left(x - \frac{\pi}{2}\right).$$

Moreover, the graph of $y = A\cot(Bx - C)$ can be obtained from the graph of the tangent function $y = A\tan(Bx - C)$ by a horizontal shift of $\frac{\pi}{2B}$ units to the right together with a vertical reflection.

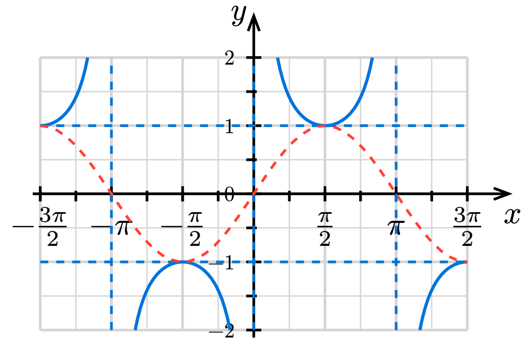
Properties of $y = A\sec(Bx - C) + D$ and $y = a\csc(Bx - C) + D$

Secant and cosecant functions are also periodic functions with vertical asymptotes.

Graphs of $y = \sec x$ with $y = \cos x$



Graphs of $y = \csc x$ with $y = \sin x$



Common Properties of $y = A\sec(Bx - C) + D$ and $y = a\csc(Bx - C) + D$

Period: $\dfrac{\pi}{2|B|}$, Range: $(-\infty, -|A| + D] \cup [|A| + D, \infty)$, Phase shift: $\dfrac{C}{B}$, Midline: $y = D$.

The graph has U-shaped and inverted-U-shaped branches that alternate between adjacent vertical asymptotes.

Properties of $y = A\sec(Bx - C) + D$

- Domain: all real numbers x such that $Bx - C \neq k\pi + \dfrac{\pi}{2}$.
- Vertical asymptotes:
$$x = \frac{k\pi}{B} + \frac{\pi + 2C}{2B}.$$
- Turning points:
$$\left(\frac{2k\pi}{B} + \frac{C}{B}, A + D\right)$$
$$\left(\frac{2k\pi}{B} + \frac{\pi + C}{B}, -A + D\right)$$

Properties of $y = a\csc(Bx - C) + D$

- Domain: all real numbers x such that $Bx - C \neq k\pi$.
- Vertical asymptotes:
$$x = \frac{k\pi}{B} + \frac{C}{B}.$$
- Turning points:
$$\left(\frac{2k\pi}{B} + \frac{\frac{\pi}{2} + C}{B}, A + D\right)$$
$$\left(\frac{2k\pi}{B} + \frac{-\frac{\pi}{2} + C}{B}, -A + D\right)$$

The period is twice the distance between two consecutive vertical asymptotes as well as the distance between two consecutive turning points with the same y-coordinate.

The midline $y = D$ is the horizontal line that has the distance $|A|$ from any turning point. Moreover, D is the average of the y-coordinates two consecutive turning points.

Graph of Cosecant Functions

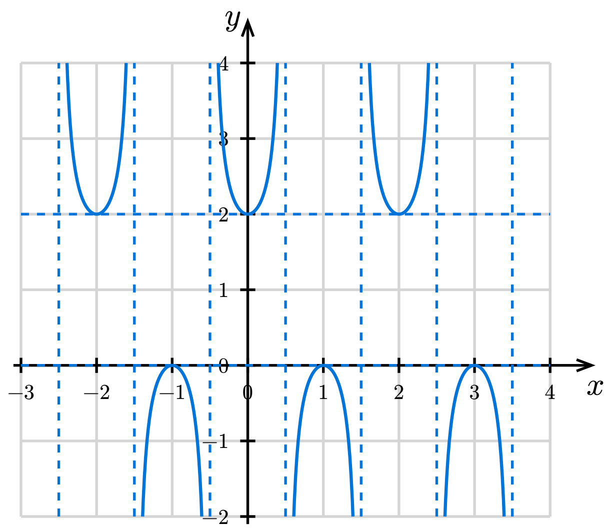
Since $\sin(x) = \cos(x - \frac{\pi}{2})$, we have $\csc(x) = \sec(x - \frac{\pi}{2})$. Therefore, the cosecant function can be obtained from the secant function by a horizontal shift of $\frac{\pi}{2B}$ units to the right.

Reciprocal Relationships

Secant and cosecant are the reciprocal functions of cosine and sine, respectively.

- Where $\cos x$ or $\sin x$ is 0, $\sec x$ or $\csc x$ has vertical asymptotes.
- Where $\cos x$ or $\sin x$ has maximum or minimum values, $\sec x$ or $\csc x$ has turning points.

Example 4.5.2. Determine the equation $y = A\sec(Bx - C) + D$ of the function defined by the following graph.



Solution. Because secant is a even function, we may assume that $B > 0$, otherwise, we can replace B by $-B$ and C by $-C$.

Because the distance between two consecutive vertical asymptotes at $x = -1$ and $x = 0$ is 1, the period is $T =$ ______, and we have

$$B = \frac{\_\_\_\_\_\_}{2} = \_\_\_\_\_\_.$$

Because the average of y-coordinates of two consecutive turning points is $\frac{2+0}{2} = 1$, so we have

$$D = \_\_\_\_\_\_ \quad \text{and the midline line is} \quad y = 1.$$

Since $(0, 2)$ is a local minimum, comparing with the standard function $y = \sec x$, we have $C = 0$ and $A > 0$.

Since the distance from a turning point $(0, 2)$ to the midline $y = 1$ is 1, we have

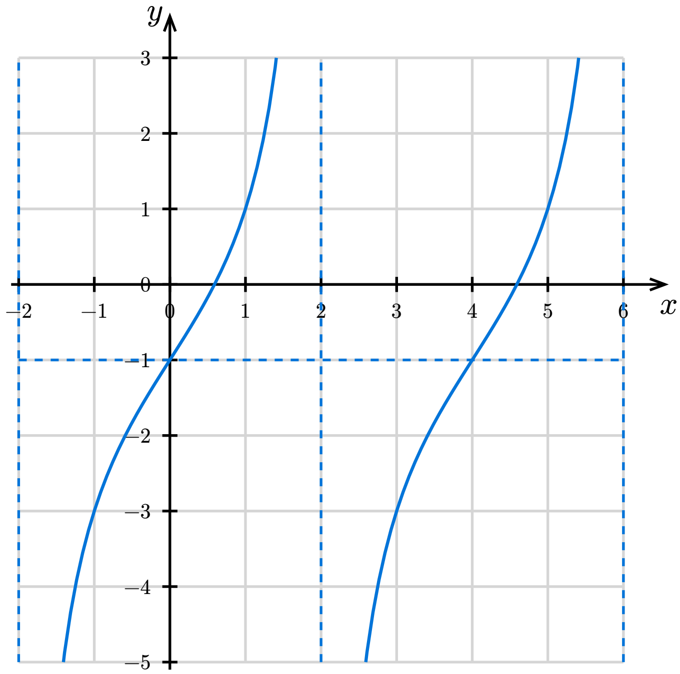
$$A = \_\_\_\_\_\_.$$

Therefore, the equation of the function is

$$y = \sec(\pi x) + 1.$$

Exercises

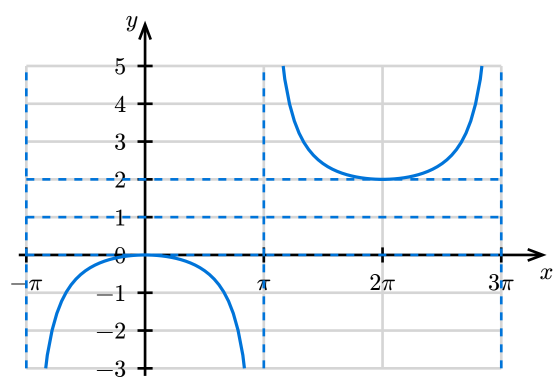
Exercise 4.5.1. Find an equation of the tangent function defined by the following graph.



Answer: $f(x) = 2\tan(\frac{\pi}{4}x) - 1$.

Exercise 4.5.2. Sketch a graph of $f(x) = -\sec\left(\frac{x}{2}\right) + 1$ in one period.

Answer:



4.6 Inverse Trigonometric Functions

Definition 4.6.1 (Inverse Functions of Trigonometric Functions)

Because the trigonometric functions are periodic and not one-to-one on their natural domains, their inverses are defined by restricting to a specific interval called the **principal branch**.

- The **inverse sine** function $y = \sin^{-1} x$ is the inverse function of the sine function $y = \sin x$ with x in $[-\frac{\pi}{2}, \frac{\pi}{2}]$. The notation $\sin^{-1} x$ is read as "sine inverse of x". The inverse sine of x is also denoted as $\arcsin x$, and read as "arcsine of x".

  The domain of $\sin^{-1} x$ is $[-1, 1]$ and its range is $[-\frac{\pi}{2}, \frac{\pi}{2}]$.

- The **inverse cosine** function $y = \cos^{-1} x$ is the inverse function of the cosine function $y = \cos x$ with x in $[0, \pi]$. The notation $\cos^{-1} x$ is read as "cosine inverse of x". The inverse cosine of x is also denoted as $\arccos x$, and read as "arccosine of x".

  The domain of $\cos^{-1} x$ is $[-1, 1]$ and its range is $[0, \pi]$.

- The **inverse tangent** function $y = \tan^{-1} x$ is the inverse function of the tangent function $y = \tan x$ with x in $(-\frac{\pi}{2}, \frac{\pi}{2})$. The notation $\tan^{-1} x$ is read as "tangent inverse of x". The inverse tangent of x is also denoted as $\arctan x$, and read as "arctangent of x".

  The domain of $\tan^{-1} x$ is $(-\infty, \infty)$ and its range is $(-\frac{\pi}{2}, \frac{\pi}{2})$.

- The **inverse cotangent** function $y = \cot^{-1} x$ is the inverse function of the cotangent function $y = \cot x$ with x in $(0, \pi)$. The notation $\cot^{-1} x$ is read as "cotangent inverse of x". The inverse cotangent of x is also denoted as $\operatorname{arccot} x$, and read as "arccotangent of x".

  The domain of $\cot^{-1} x$ is $(-\infty, \infty)$ and its range is $(0, \pi)$.

- The **inverse secant** function $y = \sec^{-1} x$ is the inverse function of the secant function $y = \sec x$ with x in $[0, \frac{\pi}{2}) \cup (\frac{\pi}{2}, \pi]$. The notation $\sec^{-1} x$ is read as "secant inverse of x". The inverse secant of x is also denoted as $\operatorname{arcsec} x$, and read as "arcsecant of x".

  The domain of $\sec^{-1} x$ is $(-\infty, -1] \cup [1, \infty)$ and its range is $[0, \frac{\pi}{2}) \cup (\frac{\pi}{2}, \pi]$.

- The **inverse cosecant** function $y = \csc^{-1} x$ is the inverse function of the cosecant function $y = \csc x$ with x in $[-\frac{\pi}{2}, 0) \cup (0, \frac{\pi}{2}]$. The notation $\csc^{-1} x$ is read as "cosecant inverse of x". The inverse cosecant of x is also denoted as $\operatorname{arccsc} x$, and read as "arccosecant of x".

  The domain of $\csc^{-1} x$ is $(-\infty, -1] \cup [1, \infty)$ and its range is $[-\frac{\pi}{2}, 0) \cup (0, \frac{\pi}{2}]$. ♣

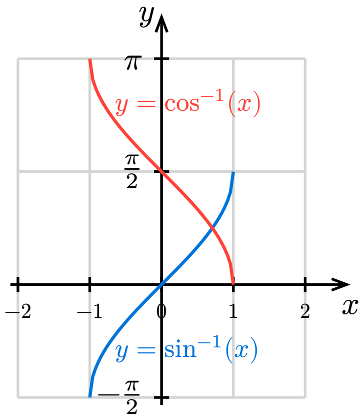
A Remark on Notations of Inverse Functions

In computer programming languages, the inverse trigonometric functions are often called by the abbreviated forms `asin`, `acos`, `atan`.

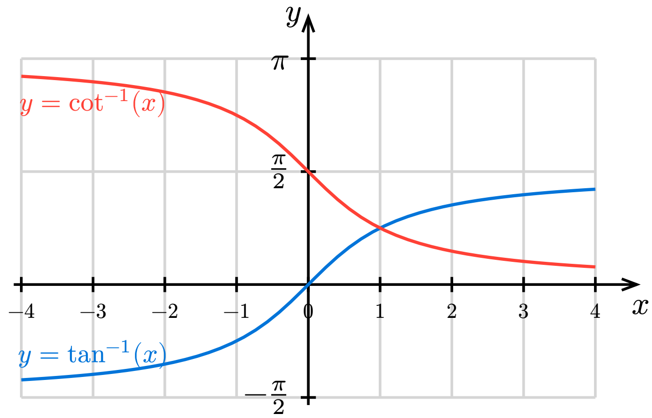
Graphs of Inverse Trigonometric Functions

The graphs of the inverse trigonometric functions can be obtained from the graphs of the corresponding trigonometric functions by reflecting about the line $y = x$.

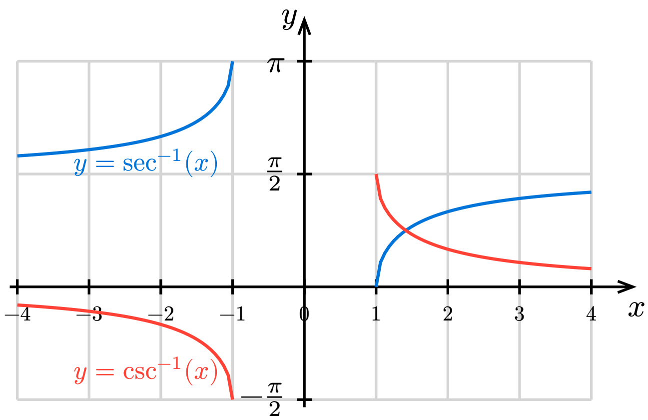
Graph of $y = \sin^{-1} x$ and $y = \cos^{-1} x$



Graph of $y = \tan^{-1} x$ and $y = \cot^{-1} x$



Graph of $y = \sec^{-1} x$ and $y = \csc^{-1} x$



Example 4.6.1. Evaluate each of the following.

1) $\sin^{-1}\left(-\frac{\sqrt{2}}{2}\right)$ 2) $\cos^{-1}\left(-\frac{\sqrt{3}}{2}\right)$ 3) $\tan^{-1}(\pi)$

Solution.

1) Because $\sin(-\frac{\pi}{4}) = -\frac{\sqrt{2}}{2}$ and $-\frac{\pi}{4}$ is in the range of $\sin^{-1} x$, we have

$$\sin^{-1}\left(-\frac{\sqrt{2}}{2}\right) = \underline{\hspace{3cm}}.$$

2) Because $\cos(5\frac{\pi}{6}) = -\frac{\sqrt{3}}{2}$ and $\frac{5\pi}{6}$ is in the range of $\cos^{-1} x$, we have

$$\cos^{-1}\left(-\frac{\sqrt{3}}{2}\right) = \underline{\hspace{3cm}}.$$

3) Since π is not a value of special angle, we use a calculator to find

$$\tan^{-1}(\pi) = \boxed{\text{2nd}} + \boxed{\text{tan}} + \boxed{\text{2nd}} + \boxed{\char`^} + \boxed{)} + \boxed{\text{enter}} = \underline{\hspace{3cm}}.$$

Composition Identities: Sine and Cosine of Inverse Trigonometric Functions

From the definition of inverse function and the Pythagorean identities, we have the following composition identities for all values of x in the domains of the corresponding inverse trigonometric functions.

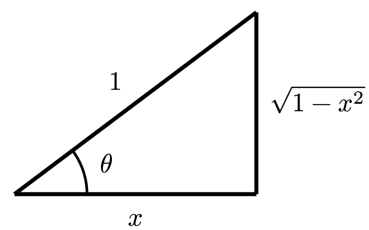
$\theta =$	$\sin^{-1} x$	$\cos^{-1} x$	$\tan^{-1} x$	$\cot^{-1} x$	$\sec^{-1} x$	$\csc^{-1} x$
$\sin\theta =$	x	$\sqrt{1-x^2}$	$\dfrac{x}{\sqrt{1+x^2}}$	$\dfrac{1}{\sqrt{1+x^2}}$	$\sqrt{1-\dfrac{1}{x^2}}$	$\dfrac{1}{x}$
$\cos\theta =$	$\sqrt{1-x^2}$	x	$\dfrac{1}{\sqrt{1+x^2}}$	$\dfrac{x}{\sqrt{1+x^2}}$	$\dfrac{1}{x}$	$\sqrt{1-\dfrac{1}{x^2}}$

Geometric Aproach to Composition Identities

The composition identities in the above box can also be verified geometrically by constructing right triangles based on the definitions of the inverse trigonometric functions. For example, to verify that $\sin(\cos^{-1} x) = \sqrt{1-x^2}$ for $-1 \le x \le 1$, we can construct a right triangle with an acute angle $\theta = \cos^{-1} x$ with adjacent side length x and hypotenuse length 1.

By the Pythagorean theorem, the opposite side to angle θ has length $\sqrt{1-x^2}$. Therefore, we have

$$\sin(\cos^{-1} x) = \sin\theta = \sqrt{1-x^2}.$$



Other Trigonometric Functions of Inverse Trigonometric Functions

Using the composition identities for sine and cosine of inverse trigonometric functions and the basic identities of trigonometric functions, we can find other trigonometric functions of inverse trigonometric functions. For example,

$$\tan(\sin^{-1} x) = \frac{\sin(\sin^{-1} x)}{\cos(\sin^{-1} x)} = \frac{x}{\sqrt{1-x^2}},$$

where $-1 < x < 1$.

Example 4.6.2. Find an exact value for $\sin(\cos^{-1}(\frac{4}{5}))$

Solution. Let $\theta = \cos^{-1}(\frac{4}{5})$. From the definition of inverse cosine function, we have

$$\cos\theta = \_\_\_\_\_ \quad \text{with} \quad 0 \le \theta \le \pi.$$

Then $\sin\theta > 0$. By Pythagorean identity, we can find $\sin\theta$ as follows:

$$\sin\theta = \sqrt{1-\left(\frac{4}{5}\right)^2} = \_\_\_\_\_$$

Example 4.6.3. Find an exact value for $\sin\left(\tan^{-1}\left(\frac{4}{7}\right)\right)$.

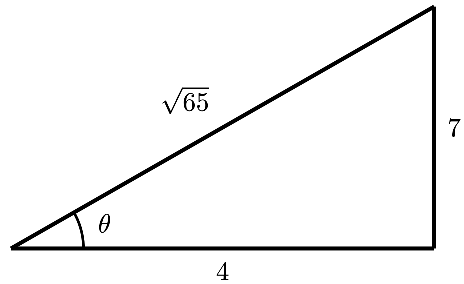
Solution. Let $\theta = \tan^{-1}\left(\frac{4}{7}\right)$. From the definition of inverse tangent function, we have

$$\tan\theta = \underline{\qquad\quad} \quad \text{with} \quad -\frac{\pi}{2} < \theta < \frac{\pi}{2}.$$

Then $\sin\theta > 0$.

We can assume that θ is an angle in a right triangle with opposite side length 4 and adjacent side length 7. By the Pythagorean theorem, the hypotenuse has length

$$\sqrt{4^2 + 7^2} = \underline{\qquad\quad}.$$



Therefore, we have

$$\sin\theta = \underline{\qquad\quad}.$$

Inverse Trigonometric Functions of Sine, Cosine and Tangent

From the definition of inverse functions, we have the following basic indentities for all values x in the ranges of the corresponding inverse trigonometric functions.

$$\sin^{-1}(\sin x) = x \quad \text{only for} \quad -\frac{\pi}{2} \le x \le \frac{\pi}{2}$$

$$\cos^{-1}(\cos x) = x \quad \text{only for} \quad 0 \le x \le \pi$$

$$\tan^{-1}(\tan x) = x \quad \text{only for} \quad -\frac{\pi}{2} < x < \frac{\pi}{2}.$$

For a general value of x, the inverse trigonometric function of its corresponding trigonometric function can be found using the reference angle. For example,

$$\sin^{-1}(\sin x) = \text{sign}(\sin x) \cdot x_{\text{ref}}$$

where $\text{sign}(\sin x)$ is the sign of $\sin x$ and x_{ref} is the reference angle of x.

Using the cofunction identities, we can obtain the following relationships between inverse sine and inverse cosine functions[5]:

$$\sin^{-1}(y) = \frac{\pi}{2} - \cos^{-1}(y) \quad \text{for} \quad -1 \le y \le 1$$

$$\cos^{-1}(y) = \frac{\pi}{2} - \sin^{-1}(y) \quad \text{for} \quad -1 \le y \le 1.$$

Consequently, we have the following identities

$$\sin^{-1}(\cos x) = \frac{\pi}{2} - x \quad \text{for} \quad 0 \le x \le \pi$$

$$\cos^{-1}(\sin x) = \frac{\pi}{2} - x \quad \text{for} \quad -\frac{\pi}{2} \le x \le \frac{\pi}{2}.$$

In genearl, there is no simple formual for other compositions of the form $f^{-1}(g(x))$ where f, g are trigonometric functions.

[5] For more relationships among inverse trigonometric functions, please refer to the wikipedia page on Inverse trigonometric functions: https://en.wikipedia.org/wiki/Inverse_trigonometric_functions.

Example 4.6.4. Evaluate the following.

1) $\sin^{-1}(\sin(\frac{\pi}{3}))$

2) $\cos^{-1}(\cos(-\frac{\pi}{3}))$

Solution.

1) Because $\frac{\pi}{3}$ is in the range $[-\frac{\pi}{2}, \frac{\pi}{2}]$ of $\sin^{-1} x$, we have

$$\sin^{-1}\left(\sin\left(\frac{\pi}{3}\right)\right) = \underline{\qquad}.$$

2) Because $-\frac{\pi}{3}$ is not in the range $[0, \pi]$ of $\cos^{-1} x$, we need to find the reference angle of $-\frac{\pi}{3}$, which is $\frac{\pi}{3}$. Since $\cos(-\frac{\pi}{3}) = \cos(\frac{\pi}{3})$ and $\frac{\pi}{3}$ is in the range of $\cos^{-1} x$, we have

$$\cos^{-1}\left(\cos\left(-\frac{\pi}{3}\right)\right) = \underline{\qquad}.$$

Example 4.6.5. Evaluate $\cos^{-1}(\sin(\frac{9\pi}{7}))$.

Solution. Because the reference angle of $\frac{9\pi}{7}$ is $\frac{2\pi}{7}$ and

$$\sin\left(\frac{9\pi}{7}\right) = -\sin\left(\frac{2\pi}{7}\right) = \sin\left(\underline{\qquad\quad}\right),$$

we have

$$\cos^{-1}\left(\sin\left(\frac{9\pi}{7}\right)\right) = \frac{\pi}{2} - \sin^{-1}\left(\sin\left(-\frac{2\pi}{7}\right)\right) = \frac{\pi}{2} - \underline{\qquad\quad}.$$

Exercises

Exercise 4.6.1. Evaluate each of the following.

1) $\sin^{-1}\left(\frac{\sqrt{3}}{2}\right)$ 2) $\cos^{-1}(-\frac{\pi}{4})$ 3) $\tan^{-1}\left(-\frac{\sqrt{3}}{3}\right)$

Answer: 1) $\frac{\pi}{3}$ 2) 2.4741 3) $-\frac{\pi}{6}$

Exercise 4.6.2. Evaluate the following.

1) $\sin^{-1}(\sin(\frac{\pi}{6}))$ 2) $\cos^{-1}(\cos(-\frac{\pi}{4}))$

Answer: 1) $\frac{\pi}{6}$ 2) $3\frac{\pi}{4}$

Exercise 4.6.3. Evaluate $\cos^{-1}\left(\sin\left(\frac{11\pi}{3}\right)\right)$.

Answer: $\frac{5\pi}{6}$.

Exercise 4.6.4. Find an exact value.

1) $\sin\left(\cos^{-1}\left(\frac{3}{5}\right)\right)$

2) $\cos\left(-\tan^{-1}\left(\frac{12}{5}\right)\right)$

Answer: 1) $\frac{4}{5}$ 2) $\frac{5}{13}$

Chapter 5 Trigonometric Identities and Equations

5.1 Simplifying Trigonometric Expressions

Basic Trigonometric Identities

Pythagorean	Quotient	Product	Negative Angle
$\sin^2 x + \cos^2 x = 1$ $1 + \tan^2 x = \sec^2 x$ $1 + \cot^2 x = \csc^2 x$	$\tan x = \dfrac{\sin x}{\cos x}$ $\cot x = \dfrac{\cos x}{\sin x}$	$\tan x \cot x = 1$ $\sin x \csc x = 1$ $\cos x \sec x = 1$	$\sin(-x) = -\sin x$ $\cos(-x) = \cos x$ $\tan(-x) = -\tan x$

Example 5.1.1. Verify the trigonometric identity.

1) $\tan\theta\cos\theta = \sin\theta$

2) $\dfrac{\sec^2\theta - 1}{\sec^2\theta} = \sin^2\theta$

Proof. We prove that the left-hand sides can be simplified to the right-hand sides.

1) Simplify the left-hand side:

$$\tan\theta\cos\theta = \frac{\sin\theta}{\underline{\qquad\qquad}}\cos\theta = \sin\theta.$$

Thus, the identity is verified.

2) Simplify the left-hand side:

$$\frac{\sec^2\theta - 1}{\sec^2\theta} = \frac{\underline{\qquad\qquad} - 1}{\frac{1}{\cos^2\theta}} = \frac{\underline{\qquad\qquad}}{\frac{1}{\cos^2\theta}} = 1 - \cos^2\theta = \underline{\qquad\qquad}.$$

Thus, the identity is verified.

Example 5.1.2. Simplify the trigonometric identity.

1) $\dfrac{\sin^2(-\theta) - \cos^2(-\theta)}{\sin(-\theta) - \cos(-\theta)}$

2) $(1 - \cos^2 x)(1 + \cot^2 x)$

Solution.

1) First apply the negative angle identities and difference of squares and then simplify:

$$\begin{aligned}\frac{\sin^2(-\theta) - \cos^2(-\theta)}{\sin(-\theta) - \cos(-\theta)} &= \frac{\sin^2(\theta) - \cos^2(\theta)}{-\sin(\theta) - \cos(\theta)} \\ &= \frac{(\sin(\theta) - \cos(\theta))(\underline{\qquad\qquad\qquad})}{-(\sin(\theta) + \cos(\theta))} \\ &= \underline{\qquad\qquad} - \underline{\qquad\qquad}.\end{aligned}$$

2) First apply the Pythagorean and Product identities and then simplify:

$$(1 - \cos^2 x)(1 + \cot^2 x) = (\underline{\qquad\qquad})\sec^2 x = \frac{\sin^2 x}{\underline{\qquad\qquad}} = \underline{\qquad\qquad}.$$

Exercises

Exercise 5.1.1. Simplify the trigonometric identity.

1) $\tan x \sin x + \sec x \cos^2 x$ 2) $\frac{\cot t + \tan t}{\sec(-t)}$ 3) $\frac{1-\cos^2 x}{\tan^2 x} + \sin^2 x$

Answer: 1) $\sec x$ 2) $\csc t$ 3) 1

5.2 Sum and Difference Angle Formulas

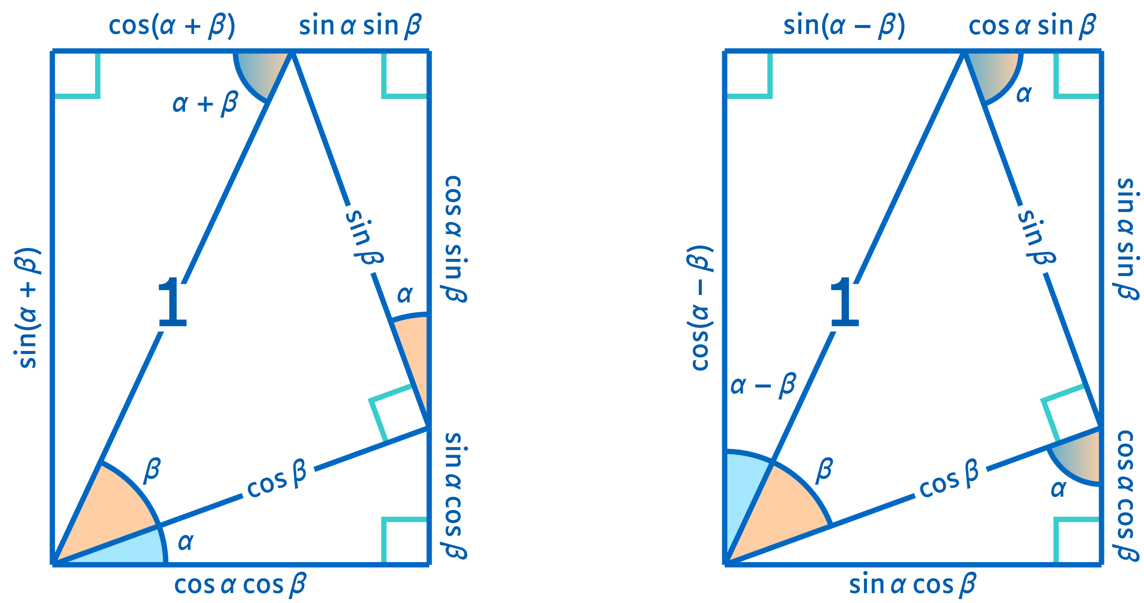
Theorem 5.2.1 (Sum and Difference Angle Formulas)

The following identities hold for all angles α and β:

$$\sin(\alpha+\beta)=\sin\alpha\cos\beta+\cos\alpha\sin\beta \qquad \sin(\alpha-\beta)=\sin\alpha\cos\beta-\cos\alpha\sin\beta$$
$$\cos(\alpha+\beta)=\cos\alpha\cos\beta-\sin\alpha\sin\beta \qquad \cos(\alpha-\beta)=\cos\alpha\cos\beta+\sin\alpha\sin\beta$$

♡

Proof. We first prove the formulas under the assumption that α, β and $\alpha+\beta$ are all in the first quadrant. In this case, the formulas follows from the following figures[6].



For other cases, these formulas can be deduced from the first quadrant case using symmetry, cofunction identities, and the reference angles. □

Remark

The sum and difference angle formulas implies the following identities which can also obtained using the unit circle and reference angles.

Cofunction	Supplementary Angle	Half Period Shifting
$\sin\left(\frac{\pi}{2}-x\right)=\cos x$	$\sin(\pi-x)=\sin x$	$\sin(x\pm\pi)=-\sin x$
$\cos\left(\frac{\pi}{2}-x\right)=\sin x$	$\cos(\pi-x)=-\cos x$	$\cos(x\pm\pi)=-\cos x$
$\tan\left(\frac{\pi}{2}-x\right)=\cot x$	$\tan(\pi-x)=-\tan x$	$\tan(x\pm\pi)=\tan x$

[6] This proof can be found in R. B. Nelsen, Proofs Without Words II, MAA, 2000, p. 46. See also the Wikipedia page https://en.wikipedia.org/wiki/List_of_trigonometric_identities#Angle_sum_and_difference_identities.

Example 5.2.1. Find the exact value.

1) $\cos(75°)$

2) $\sin(-\frac{7\pi}{12})$

Solution.

1) We can use the sum angle formula with $\alpha = 45°$ and $\beta = 30°$:

$$\begin{aligned}\cos(75°) &= \cos(45° + 30°)\\ &= \cos 45° \cdot \underline{\hspace{2cm}} - \underline{\hspace{2cm}} \cdot \sin 30°\\ &= \underline{\hspace{1.5cm}} \cdot \frac{\sqrt{3}}{2} - \frac{\sqrt{2}}{2} \cdot \underline{\hspace{1.5cm}}\\ &= \frac{\sqrt{6}-\sqrt{2}}{4}.\end{aligned}$$

2) We can use the difference angle formula with $\alpha = \frac{\pi}{3}$ and $\beta = \frac{\pi}{4}$:

$$\begin{aligned}\sin\left(-\frac{7\pi}{12}\right) &= -\sin\left(\frac{7\pi}{12}\right)\\ &= -\sin\left(\frac{\pi}{3}+\frac{\pi}{4}\right)\\ &= -\left(\sin\left(\frac{\pi}{3}\right)\cos\left(\frac{\pi}{4}\right)+\cos\left(\frac{\pi}{3}\right)\sin\left(\frac{\pi}{4}\right)\right)\\ &= -(\underline{\hspace{1.5cm}} \cdot \underline{\hspace{1.5cm}} + \underline{\hspace{1.5cm}} \cdot \underline{\hspace{1.5cm}})\\ &= -\frac{\sqrt{6}+\sqrt{2}}{4}.\end{aligned}$$

Example 5.2.2. Find the exact value of $\sin(\cos^{-1}(\frac{1}{2}) + \sin^{-1}(\frac{3}{5}))$.

Solution. Let $\alpha = \cos^{-1}(\frac{1}{2})$ and $\beta = \sin^{-1}(\frac{3}{5})$. Then we have

$$\cos\alpha = \frac{1}{2} \quad \text{and} \quad \sin\beta = \frac{3}{5}.$$

Because both $\sin(\cos^{-1} x)$ and $\cos(\sin^{-1} x)$ are positive for x in their domains, using the Pythagorean identity, we get

$$\sin\alpha = \sqrt{1-\left(\frac{1}{2}\right)^2} = \underline{\hspace{1.5cm}} \quad \text{and} \quad \cos\beta = \sqrt{1-\left(\frac{3}{5}\right)^2} = \underline{\hspace{1.5cm}}.$$

From the sum angle formula for sine, we have

$$\begin{aligned}\sin(\alpha+\beta) &= \sin\alpha\cos\beta + \cos\alpha\sin\beta\\ &= (\underline{\hspace{1.5cm}} \cdot \underline{\hspace{1.5cm}} + \underline{\hspace{1.5cm}} \cdot \underline{\hspace{1.5cm}})\\ &= \frac{27+4\sqrt{3}}{20}.\end{aligned}$$

Example 5.2.3. Given $\sin\alpha = \frac{3}{5}$, $\quad 0 < \alpha < \frac{\pi}{2}$, and $\cos\beta = -\frac{5}{13}$, $\quad \pi < \beta < \frac{3\pi}{2}$, find

1) $\sin(\alpha+\beta)$ 2) $\cos(\alpha-\beta)$ 3) $\tan(\alpha+\beta)$ 4) $\csc(\alpha-\beta)$

Solution. We first find $\cos\alpha$ and $\sin\beta$ using the Pythagorean identity:

$$\cos\alpha = \sqrt{1-\sin^2\alpha} = \sqrt{1-\left(\frac{3}{5}\right)^2} = \underline{\qquad},$$

$$\sin\beta = -\sqrt{1-\cos^2\beta} = -\sqrt{1-\left(-\frac{5}{13}\right)^2} = \underline{\qquad}.$$

1) Using the sum angle formula for sine, we have

$$\begin{aligned}\sin(\alpha+\beta) &= \sin\alpha\cos\beta + \cos\alpha\sin\beta \\ &= \frac{3}{5}\cdot\left(-\frac{5}{13}\right) + \underline{\qquad}\cdot\underline{\qquad} \\ &= \underline{\qquad}.\end{aligned}$$

2) Using the difference angle formula for cosine, we have

$$\begin{aligned}\cos(\alpha-\beta) &= \cos\alpha\cos\beta + \sin\alpha\sin\beta \\ &= \underline{\qquad}\cdot\left(-\frac{5}{13}\right) + \frac{3}{5}\cdot\underline{\qquad} \\ &= \underline{\qquad}.\end{aligned}$$

3) Because we already found $\sin(\alpha+\beta)$, to find $\tan(\alpha+\beta)$, we find $\cos(\alpha+\beta)$ and then apply the Quotient identity.

$$\begin{aligned}\cos(\alpha+\beta) &= \cos\alpha\cos\beta - \sin\alpha\sin\beta \\ &= \underline{\qquad}\cdot\left(-\frac{5}{13}\right) - \frac{3}{5}\cdot\underline{\qquad} \\ &= \underline{\qquad}.\end{aligned}$$

Thus,

$$\tan(\alpha+\beta) = \frac{\sin(\alpha+\beta)}{\cos(\alpha+\beta)} = \underline{\qquad}.$$

4) Using the difference angle formula for sine, we have

$$\begin{aligned}\sin(\alpha-\beta) &= \sin\alpha\cos\beta - \cos\alpha\sin\beta \\ &= \frac{3}{5}\cdot\left(-\frac{5}{13}\right) - \underline{\qquad}\cdot\underline{\qquad} \\ &= \underline{\qquad}.\end{aligned}$$

Thus,

$$\csc(\alpha-\beta) = \frac{1}{\sin(\alpha-\beta)} = \underline{\qquad}.$$

Exercises

Exercise 5.2.1. Find the exact value.

1) $\sin\left(-\frac{5\pi}{12}\right)$

2) $\cos\left(\frac{17\pi}{12}\right)$

Answer: 1) $-\frac{\sqrt{6}+\sqrt{2}}{4}$ 2) $-\frac{\sqrt{6}-\sqrt{2}}{4}$

Exercise 5.2.2. Find the exact value of $\cos\left(\cos^{-1}\left(\frac{1}{3}\right)-\sin^{-1}\left(\frac{4}{5}\right)\right)$.

Answer: $\frac{3+8\sqrt{2}}{15}$

Exercise 5.2.3. Given $\sin\alpha = -\frac{4}{5}, \quad \pi < \alpha < \frac{3\pi}{2}$, and $\cos\beta = \frac{12}{13}, \quad 0 < \beta < \frac{\pi}{2}$, find

1) $\sin(\alpha - \beta)$ 2) $\cos(\alpha + \beta)$ 3) $\cot(\alpha - \beta)$

Answer: 1) $-\frac{33}{65}$ 2) $-\frac{16}{65}$ 3) $\frac{56}{33}$

Exercise 5.2.4. Verify the identity

$$\sin(\alpha + \beta) + \sin(\alpha - \beta) = 2\sin\alpha\cos\beta.$$

5.3 Double/Half Angle Formulas

Theorem 5.3.1 (Double and Half Angle Formulas)

Double Angle Formulas

$$\sin(2\alpha) = 2\sin\alpha\cos\alpha \qquad \cos(2\alpha) = \cos^2\alpha - \sin^2\alpha$$

Half Angle Formulas

$$\sin^2\left(\frac{\theta}{2}\right) = \frac{1-\cos\theta}{2} \qquad \cos^2\left(\frac{\theta}{2}\right) = \frac{1+\cos\theta}{2}$$

♡

Proof. The double angle formulas follow directly from the sum angle formulas by setting $\beta = \alpha$. The half angle formulas can be derived from the double angle formulas by solving for $\sin^2(\frac{\theta}{2})$ and $\cos^2(\frac{\theta}{2})$ in the double angle formulas after replacing α by $\frac{\theta}{2}$. We leave the details to the reader. □

Other Forms of the Double Angle Formula for Cosine

From the Pythagorean identity, we have the following equivalent forms of the double angle formula for cosine:

$$\begin{aligned}\cos(2\alpha) &= 2\cos^2\alpha - 1\\ &= 1 - 2\sin^2\alpha.\end{aligned}$$

Example 5.3.1. Find $\sin 15°$ and $\cos 15°$.

Solution. We can use the half angle formulas with $\theta = 30°$:

$$\sin 15° = \sin\left(\frac{30°}{2}\right) = \sqrt{\frac{1-\cos 30°}{2}} = \sqrt{\frac{\underline{\qquad\qquad}}{4}} = \frac{\sqrt{2-\sqrt{3}}}{2},$$

$$\cos 15° = \cos\left(\frac{30°}{2}\right) = \sqrt{\frac{1+\cos 30°}{2}} = \sqrt{\frac{\underline{\qquad\qquad}}{4}} = \frac{\sqrt{\underline{\qquad\qquad}}}{2}.$$

Remark

Note that we have already found $\cos 15°$ in the previous section using the sum angle formula. The result here is consistent with the one previously obtained because

$$2+\sqrt{3} = \frac{3}{2} - 2\sqrt{\frac{3}{2}}\sqrt{\frac{1}{2}} + \frac{1}{2} = \left(\sqrt{\frac{3}{2}} + \sqrt{\frac{1}{2}}\right)^2 = \left(\frac{\sqrt{6}+\sqrt{2}}{2}\right)^2$$

and thus

$$\frac{\sqrt{2+\sqrt{3}}}{2} = \frac{\sqrt{\left(\frac{\sqrt{6}+\sqrt{2}}{2}\right)^2}}{2} = \frac{\sqrt{6}+\sqrt{2}}{4}.$$

Example 5.3.2. Find the exact value.

1) $\sin\left(2\cos^{-1}\left(\frac{3}{5}\right)\right)$

2) $\tan\left(2\sin^{-1}\left(\frac{3}{5}\right)\right)$

Solution. Let $\alpha = \cos^{-1}\left(\frac{3}{5}\right)$. Then we have

$$\cos\alpha = \frac{3}{5}.$$

Using the Pythagorean identity, we get

$$\sin\alpha = \sqrt{1-\left(\frac{3}{5}\right)^2} = \underline{\qquad}.$$

From the double angle formula for sine, we have

$$\begin{aligned}\sin(2\alpha) &= 2\sin\alpha\cos\alpha \\ &= 2\,\underline{\qquad}\cdot\frac{3}{5} \\ &= \frac{24}{25}.\end{aligned}$$

Let $\beta = \sin^{-1}\left(\frac{3}{5}\right)$. Then we have

$$\sin\beta = \frac{3}{5}.$$

Using the Pythagorean identity, we get

$$\cos\beta = \sqrt{1-\left(\frac{3}{5}\right)^2} = \underline{\qquad}.$$

From the double angle formulas for sine and cosine and the quotient formular for tangent, we have

$$\tan(2\beta) = \frac{\sin(2\beta)}{\cos(2\beta)} = \frac{2\sin\beta\cos\beta}{\cos^2\beta-\sin^2\beta} = \frac{2\cdot\frac{3}{5}\cdot\underline{\qquad}}{\left(\underline{\qquad}\right)^2-\left(\frac{3}{5}\right)^2} = \frac{24}{7}.$$

Example 5.3.3. Given that $\tan\alpha = \frac{8}{15}$ and α lies in quadrant III, find the exact value of the following:

1) $\sin\left(\frac{\alpha}{2}\right)$

2) $\cos\left(\frac{\alpha}{2}\right)$

3) $\tan\left(\frac{\alpha}{2}\right)$

Solution. Because α lies in quadrant III, $\pi < \alpha < \frac{3\pi}{2}$ and hence $\frac{\pi}{2} < \frac{\alpha}{2} < \frac{3\pi}{4} < \pi$. Therefore, both $\sin\alpha$ and $\cos\alpha$ are negative, $\sin\left(\frac{\alpha}{2}\right)$ is positive and $\cos\left(\frac{\alpha}{2}\right)$ is negative.

Using the Pythagorean identity and the Quotient identity, we have

$$\cos\alpha = -\sqrt{\frac{1}{1+\tan^2\alpha}} = -\sqrt{\frac{1}{1+\left(\frac{8}{15}\right)^2}} = \underline{\qquad},$$

$$\sin\alpha = \tan\alpha\cdot\cos\alpha = \frac{8}{15}\cdot\left(-\frac{15}{17}\right) = \underline{\qquad}.$$

1) From the half angle formula for sine, we have

$$\sin\left(\frac{\alpha}{2}\right) = \sqrt{\frac{1-\cos\alpha}{2}} = \sqrt{\frac{1-\left(-\frac{15}{17}\right)}{2}} = \sqrt{\underline{\qquad}} = \frac{4\sqrt{17}}{17}.$$

2) From the half angle formula for cosine, we have

$$\cos\left(\frac{\alpha}{2}\right) = -\sqrt{\frac{1+\cos\alpha}{2}} = -\sqrt{\frac{1+\left(-\frac{15}{17}\right)}{2}} = -\sqrt{\underline{\qquad}} = -\frac{\sqrt{17}}{17}.$$

3) From the half angle formula for tangent, we have

$$\tan\left(\frac{\alpha}{2}\right) = \frac{\sin\frac{\alpha}{2}}{\cos\frac{\alpha}{2}} = \underline{\qquad}.$$

Double and Half Angle Formulas for Tangent

From the double and half angle formulas for sine and cosine and the Quotient identity, we have the following formulas for tangent:

$$\tan(2\alpha) = \frac{2\tan\alpha}{1-\tan^2\alpha},$$

$$\tan\left(\frac{\alpha}{2}\right) = \frac{\sin\alpha}{1+\cos\alpha} = \frac{1-\cos\alpha}{\sin\alpha}.$$

In later sections, we will solve equations involving powers of sine and cosine. The following example shows how to use the double and half angle formulas to rewrite such expressions into equivalent expressions without powers greater than 1.

Example 5.3.4. Write the expression into an equivalent expression without any powers greater than 1.

1) $\cos^4 x$

2) $\sin^3 x \cos x$

Solution.

1) Using the half angle formula for cosine twice, we have

$$\begin{aligned}\cos^4 x &= (\cos^2 x)^2 \\ &= \left(\frac{1+\cos(2x)}{2}\right)^2 \\ &= \frac{1}{4}(1+2\cos(2x)+\cos^2(2x)) \\ &= \frac{1}{4}\left(1+2\cos(2x)+\frac{\underline{\qquad}}{2}\right) \\ &= \underline{\qquad\qquad}.\end{aligned}$$

2) Using the double angle formula for sine and the half angle formula for cosine, we have

$$\begin{aligned}\sin^3 x\cos x &= \sin^2 x \cdot (\sin x \cdot \cos x) \\ &= \frac{1-\cos(2x)}{2} \cdot \frac{\underline{\qquad}}{2} \\ &= \frac{1}{4} \cdot \underline{\qquad} - \frac{1}{4}\sin(2x)\cos(2x) \\ &= \underline{\qquad\qquad}.\end{aligned}$$

Exercises

Exercise 5.3.1. Find the exact value.

1) $\cos\left(2\sin^{-1}\left(\frac{4}{5}\right)\right)$

2) $\tan\left(2\cos^{-1}\left(\frac{4}{5}\right)\right)$

Answer: 1) $-\frac{7}{25}$ 2) $\frac{24}{7}$

Exercise 5.3.2. Given that $\sin\alpha = -\frac{4}{5}$ and α lies in quadrant IV, find the exact value of $\tan\left(\frac{\alpha}{2}\right)$.

Answer: $-\frac{1}{2}$

Exercise 5.3.3. Rewrite the expression with no exponent higher than 1 and no product of two trigonometric functions.

1) $8\sin^4\left(\frac{3x}{2}\right)$

2) $4\cos^3(x)\sin x.$

Answer: 1) $\frac{3}{8}-\frac{1}{2}\cos(3x)+\frac{1}{8}\cos(6x)$ 2) $\frac{3}{4}\sin(2x)+\frac{1}{4}\sin(4x)$

5.4 Sum-to-Product and Product-to-Sum Formulas

Theorem 5.4.1 (Product-to-Sum and Sum-to-Product Identities)

Product-to-Sum Identities

$$\cos\alpha\cos\beta = \frac{1}{2}(\cos(\alpha-\beta)+\cos(\alpha+\beta))$$

$$\sin\alpha\cos\beta = \frac{1}{2}(\sin(\alpha+\beta)+\sin(\alpha-\beta))$$

$$\sin\alpha\sin\beta = \frac{1}{2}(\cos(\alpha-\beta)-\cos(\alpha+\beta))$$

Sum-to-Product Identities

$$\cos\alpha+\cos\beta = 2\cos\left(\frac{\alpha+\beta}{2}\right)\cos\left(\frac{\alpha-\beta}{2}\right)$$

$$\sin\alpha+\sin\beta = 2\sin\left(\frac{\alpha+\beta}{2}\right)\cos\left(\frac{\alpha-\beta}{2}\right)$$

$$\cos\alpha-\cos\beta = -2\sin\left(\frac{\alpha+\beta}{2}\right)\sin\left(\frac{\alpha-\beta}{2}\right)$$

Proof. The product-to-sum identities can be derived by applying the sum and difference angle formulas.

The sum-to-product identities can be derived from the product-to-sum identities by replacing α and β with $\frac{\alpha+\beta}{2}$ and $\frac{\alpha-\beta}{2}$ respectively.

We leave the details to the reader. □

Example 5.4.1. Write the following product as a sum

1) $2\cos\left(\frac{7x}{2}\right)\cos\left(\frac{3x}{2}\right)$

2) $\sin(3\theta)\cos(5\theta)$

Solution.

1) Using the product-to-sum identity for cosine, we have

$$2\cos\left(\frac{7x}{2}\right)\cos\left(\frac{3x}{2}\right) = \cos\left(\frac{7x}{2}-\frac{3x}{2}\right)+\cos\left(\frac{7x}{2}+\frac{3x}{2}\right)$$

$$= \underline{\hspace{4cm}}.$$

2) Using the product-to-sum identity for sine and cosine, we have

$$\sin(3\theta)\cos(5\theta) = \frac{1}{2}(\sin(3\theta+5\theta)+\sin(3\theta-5\theta))$$

$$= \frac{1}{2}(\sin(\underline{\hspace{1.5cm}})+\sin(\underline{\hspace{1.5cm}}))$$

$$= \underline{\hspace{4cm}}.$$

Example 5.4.2. Write the following difference or sum expression as a product.

1) $\sin(3\theta) - \sin\theta$ **2)** $\cos(2\theta) + \cos(4\theta)$ **3)** $\sin\theta - \cos\theta$

Solution.

1) Using the sum-to-product identity for sine, we have

$$\begin{aligned}\sin(3\theta) - \sin\theta &= 2\cos\left(\frac{3\theta+\theta}{2}\right)\sin\left(\frac{3\theta-\theta}{2}\right)\\ &= 2\cos(2\theta)\sin(\theta).\end{aligned}$$

2) Using the sum-to-product identity for cosine, we have

$$\begin{aligned}\cos(2\theta) + \cos(4\theta) &= \underline{\qquad\qquad\qquad}\\ &= 2\cos(3\theta)\cos(-\theta)\\ &= 2\cos(3\theta)\ \underline{\qquad\quad}.\end{aligned}$$

3) Using the cofunction identity and sum-to-product identities for sine, we have

$$\begin{aligned}\sin\theta - \cos\theta &= \sin\theta - \sin(\underline{\qquad\quad})\\ &= 2\sin\left(\frac{\theta + \underline{\qquad\quad}}{2}\right)\cos\left(\frac{\theta - \underline{\qquad\quad}}{2}\right)\\ &= 2\sin\left(\frac{\pi}{4}\right)\cos\left(\frac{\underline{\qquad\quad}}{4}\right)\\ &= \underline{\qquad\qquad\qquad}.\end{aligned}$$

Exercises

Exercise 5.4.1. Write the following product as a sum

1) $\sin\left(\frac{\theta}{2}\right)\cos\left(\frac{5\theta}{2}\right)$

2) $\sin(4\theta)\sin(2\theta)$

Answer: 1) $\frac{1}{2}(\sin(3\theta) - \sin(2\theta))$ 2) $\frac{1}{2}(\cos(2\theta) - \cos(6\theta))$

Exercise 5.4.2. Write the following difference or sum expression as a product.

1) $\sin(5\theta) - \sin\theta$

2) $\cos(\theta) + \sin(\theta)$

3) $\cos(3\theta) + \cos(5\theta)$

Answer: 1) $2\cos(3\theta)\sin(2\theta)$ 2) $\sqrt{2}\sin(\theta + \frac{\pi}{4})$ 3) $2\cos(4\theta)\cos(\theta)$

Exercise 5.4.3. Find the exact value.

1) $\sin(75^\circ) - \cos(75^\circ)$

2) $\sin(15^\circ) + \sin(135^\circ)$

Answer: 1) $\frac{\sqrt{2}}{2}$ 2) $\frac{\sqrt{2}+\sqrt{6}}{2}$

5.5 Solving Trigonometric Equations

How to Solve a Trigonometric Equation

To solve a trigonometric equation of an angle θ, we typically follow these steps:

1) Use algebraic manipulation and trigonometric identities to express the equation in the standard form:
$$f(X) = c,$$
where f is a basic trigonometric function ($\sin$, $\cos$, or $\tan$), X is an expression in θ, and c is a constant. Note that the other three trigonometric functions ($\csc$, $\sec$, and $\cot$) are reciprocals of $\sin$, $\cos$, and $\tan$ respectively.

2) Apply the inverse trigonometric function to both sides to find a particular solution $X = f^{-1}(c)$ and use symmetry and periodicity to find the general solution for X:
   - If $f(X) = \sin X$, then the general solution is
   $$X = \sin^{-1}(c) + 2k\pi \quad \text{or} \quad X = \pi - \sin^{-1}(c) + 2k\pi, \quad k \in \mathbb{Z}.$$
   - If $f(X) = \cos X$, then the general solution is
   $$X = \cos^{-1}(c) + 2k\pi \quad \text{or} \quad X = -\cos^{-1}(c) + 2k\pi, \quad k \in \mathbb{Z}.$$
   - If $f(X) = \tan X$, then the general solution is
   $$X = \tan^{-1}(c) + k\pi, \quad k \in \mathbb{Z}.$$

3) Solve for θ in the specified interval from the general solution for X.

Remark

The reason that we have two forms for the general solutions of $\sin X = c$ and $\cos X = c$ lies in the symmetries of their graphs. Specifically, consider the following identities:
$$\sin\left(\frac{\pi}{2} + X\right) = \sin\left(\frac{\pi}{2} - X\right) \quad \text{and} \quad \cos(X) = \cos(-X).$$
By substituting X with $\frac{\pi}{2} - X$, the first identity becomes
$$\sin(X) = \sin(\pi - X).$$
It follows that if $X = \sin^{-1}(c)$ is a solution to $\sin X = c$, then $X = \pi - \sin^{-1}(c)$ is also a solution.

Similarly, if $X = \cos^{-1}(c)$ is a solution to $\cos X = c$, then $X = -\cos^{-1}(c)$ is also a solution.

However, for $\tan X$, the symmetries,
$$\tan\left(\frac{k\pi}{2} + X\right) = -\tan\left(\frac{k\pi}{2} - X\right) = \tan\left(X - \frac{k\pi}{2}\right),$$
implies that $\tan(X) = \tan(k\pi - X)$. Hence, any two solutions of $\tan X = c$ differ by $k\pi$ for some integer k, which is already captured in the general solution.

Example 5.5.1. Find all solutions in their exact form for the equation.

1) $\cos\theta = \frac{1}{2}$ 2) $\sin\theta = \frac{1}{2}$

Solution.

1) We apply the inverse cosine function to both sides to find a particular solution:

$$\theta = \cos^{-1}\left(\frac{1}{2}\right) = \underline{\qquad}.$$

Using the general solution for cosine, we have

$$\theta = \underline{\qquad} + 2k\pi \quad \text{or} \quad \theta = -\underline{\qquad} + 2k\pi, \quad k \in \mathbb{Z}.$$

2) We apply the inverse sine function to both sides to find a particular solution:

$$\theta = \sin^{-1}\left(\frac{1}{2}\right) = \underline{\qquad}.$$

Using the general solution for sine, we have

$$\theta = \underline{\qquad} + 2k\pi \quad \text{or} \quad \pi - \underline{\qquad} + 2k\pi = \underline{\qquad} + 2k\pi, \quad k \in \mathbb{Z}.$$

Example 5.5.2. Solve the equation exactly:

$$2\cos\theta - 3 = -5, \quad 0 \le \theta < 2\pi.$$

Solution. We first isolate the cosine function:

$$2\cos\theta = -2$$
$$\cos\theta = -1.$$

We apply the inverse cosine function to both sides to find a particular solution:

$$\theta = \cos^{-1}(-1) = \underline{\qquad}.$$

Using the general solution for cosine, we have

$$\theta = \underline{\qquad} + 2k\pi \quad \text{or} \quad \theta = -\underline{\qquad} + 2k\pi, \quad k \in \mathbb{Z}.$$

Because $0 \le \theta < 2\pi$, both forms of the general solutions lead to the same solution

$$\theta = \underline{\qquad}.$$

Example 5.5.3. Solve the equation exactly:

$$2\sin^2\theta - 1 = 0, \quad 0 \le \theta < 2\pi.$$

Solution. By the double angle formula for cosine: $\cos(2\theta) = 1 - 2\sin^2\theta$, we see that the equation is equivalent to

$$\cos(2\theta) = 0.$$

Therefore,

$$2\theta = \underline{\qquad} + 2k\pi \quad \text{or} \quad 2\theta = -\underline{\qquad} + 2k\pi$$
$$\theta = \underline{\qquad} + k\pi \quad \text{or} \quad \theta = -\underline{\qquad} + k\pi.$$

Because $0 \le \theta < 2\pi$, the solutions are

$$\theta = \underline{\qquad}, \underline{\qquad}, \underline{\qquad}, \underline{\qquad}.$$

Example 5.5.4. Solve the equation exactly:

$$4\cos^2\theta + 3\cos\theta - 1 = 0, \quad 0 \le \theta < 2\pi.$$

Solution. We first substitute $\cos\theta$ by x and solve the resulting quadratic equation:

$$\begin{aligned} 4x^2 + 3x - 1 &= 0 \\ (4x-1)(\underline{\qquad\qquad}) &= 0 \\ 4x - 1 = 0 \quad \text{or} \quad \underline{\qquad\qquad} &= 0 \\ x = \frac{1}{4} \quad \text{or} \quad\quad x &= -1. \end{aligned}$$

We now solve for θ in each case.

1) When $\cos\theta = \frac{1}{4}$, we apply the inverse cosine function to both sides to find a particular solution:

$$\theta = \cos^{-1}\left(\frac{1}{4}\right) = \underline{\qquad\quad}.$$

Using the general solution for cosine, we have

$$\theta = \underline{\qquad\quad} + 2k\pi \quad \text{or} \quad \theta = -\underline{\qquad\quad} + 2k\pi, \quad k \in \mathbb{Z}.$$

Because $0 \le \theta < 2\pi$, the solutions are

$$\theta = \underline{\qquad\quad}, \quad \underline{\qquad\quad}.$$

2) When $\cos\theta = -1$, we apply the inverse cosine function to both sides to find a particular solution:

$$\theta = \cos^{-1}(-1) = \underline{\qquad\quad}.$$

Using the general solution for cosine, we have

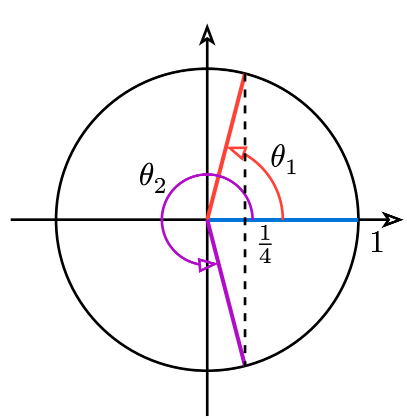
$$\theta = \underline{\qquad\quad} + 2k\pi \quad \text{or} \quad \theta = -\underline{\qquad\quad} + 2k\pi, \quad k \in \mathbb{Z}.$$

Because $0 \le \theta < 2\pi$, there is only one solution and the solution is

$$\theta = \underline{\qquad\quad}.$$

Remark

In the previous example, because $0 \le \theta \le 2\pi$, we could directly determine θ from the unit circle. For example, for $\cos\theta = \frac{1}{4}$, the figure on the right shows two solutions: $\theta_1 = \cos^{-1}\left(\frac{1}{4}\right)$ and $\theta_2 = 2\pi - \cos^{-1}\left(\frac{1}{4}\right)$.



However, in general, if the equation involves a multiple angle, such as $\cos(2\theta)$ or $\sin(3\theta)$. In such cases, it is better to rely on the general solution approach as the unit circle may not directly provide all solutions in the specified interval.

Example 5.5.5. Solve the equation exactly: $2\cos^2\theta - 3\sin\theta = 3$.

Solution. By Pythagorean identity, $\cos^2\theta = 1 - \sin^2\theta$, we can solve for $\sin\theta$ as follows:

$$\begin{aligned} 2(1-\sin^2\theta) - 3\sin\theta &= 3 \\ -2\sin^2\theta - 3\sin\theta + 2 &= 0 \\ 2\sin^2\theta + 3\sin\theta - 2 &= 0 \\ (\underline{\qquad\qquad})(\sin\theta + 2) &= 0 \\ \underline{\qquad\qquad} = 0 \quad &\text{or} \quad \sin\theta + 2 = 0 \\ \sin\theta = \underline{\qquad} \quad &\text{or} \quad \sin\theta = -2 \end{aligned}$$

The equation $\sin\theta = -2$ has no solution because the range of the sine function is $[-1, 1]$. For the equation

$$\sin\theta = \frac{1}{2},$$

applying the inverse sine function to both sides gives a particular solution:

$$\theta = \sin^{-1}\left(\frac{1}{2}\right) = \underline{\qquad}.$$

The following general solutions are

$$\theta = \underline{\qquad} + 2k\pi \quad \text{or} \quad \theta = \underline{\qquad} + 2k\pi.$$

Example 5.5.6. Solve the equation exactly:

$$\cos x\cos(2x) - \sin x\sin(2x) = \frac{\sqrt{3}}{2}, \quad 0 \le x < \pi.$$

Solution. By the sum angle formula for cosine, we have

$$\cos x\cos(2x) - \sin x\sin(2x) = \cos(x + 2x) = \cos(3x).$$

Therefore, the equation is equivalent to

$$\cos(3x) = \frac{\sqrt{3}}{2}.$$

We apply the inverse cosine function to both sides to find a particular solution:

$$3x = \cos^{-1}\left(\frac{\sqrt{3}}{2}\right) = \underline{\qquad}.$$

Using the general solution for cosine, we have

$$3x = \underline{\qquad} + 2k\pi \quad \text{or} \quad 3x = -\underline{\qquad} + 2k\pi, \quad k \in \mathbb{Z}.$$

Dividing both sides by 3, we get

$$x = \frac{\underline{\qquad}}{3} + \frac{2k\pi}{3} \quad \text{or} \quad x = -\frac{\underline{\qquad}}{3} + \frac{2k\pi}{3}.$$

Because $0 \le x < 2\pi$, the solutions are

$$x = \underline{\qquad}, \quad \underline{\qquad}, \quad \underline{\qquad}.$$

Example 5.5.7. Solve the equation exactly:

$$\cos(3\theta) = \cos\theta, \quad \text{where} \quad 0 \le \theta < \pi.$$

Solution. By the sum-to-product identity for cosine, we have

$$\begin{aligned} \cos(3\theta) - \cos\theta &= -2\sin\left(\frac{3\theta+\theta}{2}\right)\sin\left(\frac{3\theta-\theta}{2}\right) \\ &= -2\sin(2\theta)\sin(\theta). \end{aligned}$$

Therefore, the equation is equivalent to

$$-2\sin(2\theta)\sin(\theta) = 0$$

$$\sin(2\theta) = 0 \quad \text{or} \quad \sin(\theta) = 0.$$

1) For the equation $\sin(2\theta) = 0$, we apply the inverse sine function to both sides to find a particular solution:

$$2\theta = \sin^{-1}(0) = \underline{\qquad}.$$

Using the general solution for sine, we have

$$2\theta = \underline{\qquad} + 2k\pi \quad \text{or} \quad 2\theta = \underline{\qquad} + 2k\pi.$$

Dividing both sides by 2, we get

$$\theta = \underline{\qquad} + k\pi \quad \text{or} \quad \theta = \underline{\qquad} + k\pi.$$

Because $0 \le \theta < \pi$, the solutions are

$$\theta = \underline{\qquad}, \ \underline{\qquad}.$$

2) For the equation $\sin(\theta) = 0$, we apply the inverse sine function to both sides to find a particular solution:

$$\theta = \sin^{-1}(0) = \underline{\qquad}.$$

Using the general solution for sine, we have

$$\theta = \underline{\qquad} + 2k\pi \quad \text{or} \quad \theta = \underline{\qquad} + 2k\pi.$$

Because $0 \le \theta < \pi$, there is only one solution and the solution is

$$\theta = \underline{\qquad}.$$

Therefore, the solutions to the original equation are

$$\theta = \underline{\qquad}, \ \underline{\qquad}, \ \underline{\qquad}.$$

Exercises

Exercise 5.5.1. Solve the equation exactly:

$$4\sin\theta\cos\theta - \sqrt{3} = 0, \quad 0 \le \theta < 2\pi.$$

Answer: $\theta = \frac{\pi}{6}, \ \frac{\pi}{3}, \ \frac{7\pi}{6}, \ \frac{4\pi}{3}.$

Exercise 5.5.2. Solve the equation exactly:

$$\cos^2\theta - 2\cos\theta - 3 = 0, \quad 0 \le \theta < 2\pi.$$

Answer: $\theta = \pi.$

Exercise 5.5.3. Solve the equation exactly over the given interval:

$$2\cos^2\theta - 9\sin\theta + 3 = 0, \quad 0 \le \theta < 2\pi.$$

Answer: $\theta = \frac{\pi}{6}, \frac{5\pi}{6}$.

Exercise 5.5.4. Solve the equation exactly over the given interval:

$$\sin x\cos(2x) + \cos x\sin(2x) = \frac{1}{2}, \quad 0 \le x < \pi.$$

Answer: $x = \frac{\pi}{18}, \quad \frac{5\pi}{18}, \quad \frac{13\pi}{18}, \quad \frac{17\pi}{18}$.

Chapter 6 Laws of Sines and Cosines

6.1 Law of Sines

Theorem 6.1.1 (Law of Sines)

Given a triangle $\triangle ABC$ with sides of lengths a, b, and c opposite to angles A, B, and C, respectively, then

$$\frac{\sin A}{a} = \frac{\sin B}{b} = \frac{\sin C}{c} \quad \text{or} \quad \frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$$

♡

Proof. It follows from the following theorem of area of triangle using SAS (side-angle-side).

□

Theorem 6.1.2 (Area of Triangle Using SAS)

Given a triangle $\triangle ABC$ with sides of lengths a, b, and c opposite to angles A, B, and C, respectively, then the area S of the triangle is

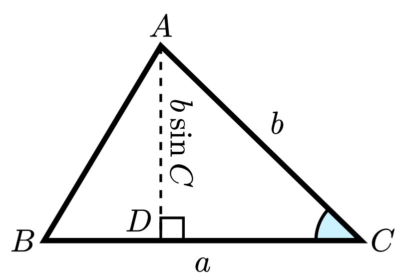
$$S = \frac{1}{2}ab\sin C = \frac{1}{2}ac\sin b = \frac{1}{2}bc\sin A.$$

♡

Proof. We proof the first formula only. Other two formulas can be proved similarly.

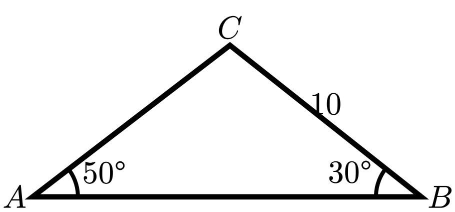
Drop a perpendicular line from vertex B to side AC at point D. Then

$$S = \frac{1}{2}\,\text{base}\cdot\text{height} = \frac{1}{2}a\cdot h = \frac{1}{2}a\cdot b\sin C.$$



□

Example 6.1.1. Solve for the unknown side and angles. Round your answers to the nearest tenth.



Solution. Because the sum of the angles in a triangle is $180°$, we have

$$C = 180° - 50° - 30° = \underline{\qquad}.$$

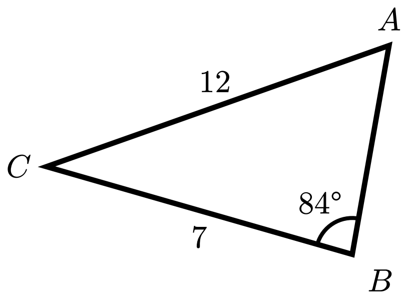
Using the Law of Sines, we have

$$\frac{AC}{\sin 30°} = \frac{AB}{\sin 100°} = \frac{10}{\sin 50°}.$$

Thus,

$$AB = \frac{\underline{\qquad}\cdot 10}{\sin 50°} \approx 7.66 \quad \text{and} \quad b = \frac{\underline{\qquad}\cdot 10}{\sin 50°} \approx 5.13.$$

Example 6.1.2. Solve for the unknown side and angles. Round your answers to the nearest tenth.



Solution. We first find the angle A. Using the Law of Sines, we have

$$\frac{\sin A}{7} = \frac{\sin 84°}{12}$$
$$\sin A = \frac{\underline{\qquad\quad} \cdot \sin 84°}{12}$$

Because both A and C are acute angles as shown in the figure, we have

$$A = \sin^{-1}\left(\frac{7 \sin 84°}{12}\right) \approx \underline{\qquad\quad}.$$

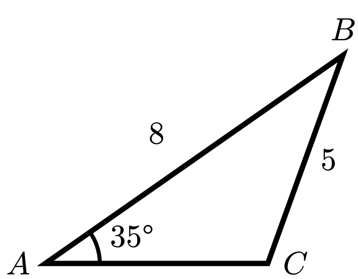
Thus,

$$C = 180° - 84° - A \approx \underline{\qquad\quad}.$$

Apply the Law of Sines again to find side AB:

$$\frac{AB}{\sin 84°} = \frac{7}{\sin A}$$
$$AB = \frac{\underline{\qquad\quad} \cdot \sin 84°}{\sin A} \approx \underline{\qquad\quad}.$$

Example 6.1.3. Solve for the unknown side and angles. Round your answers to the nearest tenth.



Solution. We first find the angle C. Using the Law of Sines, we have

$$\frac{\sin C}{8} = \frac{5}{\sin 35°}$$
$$\sin C = \frac{\underline{\qquad\quad}}{\sin 35°}$$

Thus,

$$C = \pi - \sin^{-1}\left(\underline{\qquad\quad}\right) \approx \underline{\qquad\quad}, \quad \text{and } B = 180° - 35° - C \approx \underline{\qquad\quad}.$$

Apply the Law of Sines again to find side AC:

$$\frac{AC}{\sin B} = \frac{5}{\sin 35°}$$
$$AC = \frac{\underline{\qquad\quad} \cdot \sin B}{\sin 35°} \approx \underline{\qquad\quad}.$$

⚠ Be Aware of the Angle Measurement Unit

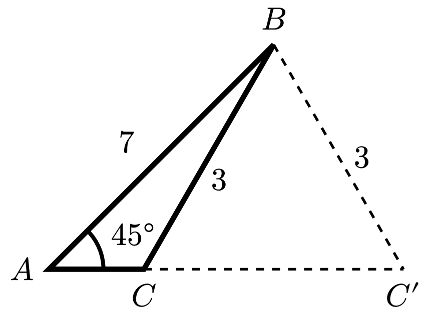
When calculating the angle using the inverse sine function, be aware of the measurement unit (degree vs. radian) set on your calculator.

Example 6.1.4. Find all possible triangles if one side has length 3 opposite an angle of $45°$, and a second side has length 7.

Solution.

The angle measured $45°$ is A, the side of length 7 is AB, and the side opposite to A is BC. Using the Law of Sines, we have

$$\frac{\sin C}{7} = \frac{\sin 45°}{3}$$
$$\sin B = \frac{\underline{\hspace{2cm}} \cdot \sin 45°}{3}.$$



Note that as an angle of a triangle, $0 < B < 180°$. Thus, as shown in the figure above, there are two possible positions for the side BC, and hence two possible values α and β as shown in the figure above for angle B:

$$\angle ABC = \sin^{-1}(\underline{\hspace{2cm}}) \approx \underline{\hspace{2cm}}, \quad \text{and} \quad \angle ABC' = 180° - \angle ABC \approx \underline{\hspace{2cm}}.$$

For angle $\angle ABC$, we have

$$\angle ACB = 180° - 45° - \angle ABC \approx \underline{\hspace{2cm}}.$$

Using the Law of Sines again, we have

$$BC = \frac{\underline{\hspace{2cm}} \cdot 3}{\sin 45°} \approx \underline{\hspace{2cm}}.$$

For angle $\angle ABC'$, we have

$$\angle AC'B = 180° - 45° - \angle ABC' \approx \underline{\hspace{2cm}}.$$

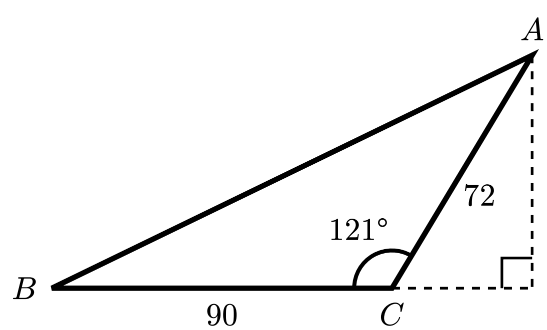
Using the Law of Sines again, we have

$$BC' = \frac{\underline{\hspace{2cm}} \cdot 3}{\sin 45°} \approx \underline{\hspace{2cm}}.$$

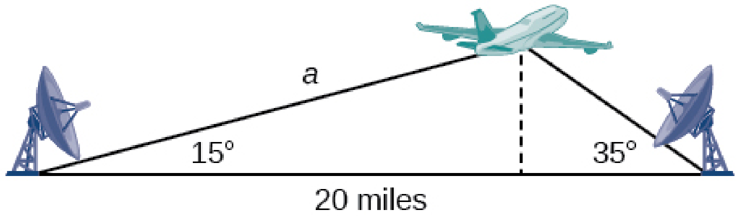
Example 6.1.5. Find the area of a triangle with sides $a = 90$, $b = 72$, and the angle $C = 121°$ formed by those two sides. Round the area to the nearest integer.

Solution. By the area formula for a triangle with known side, angle, side, the area is

$$\begin{aligned} S &= \frac{1}{2} a \cdot b \cdot \sin C \\ &= \frac{1}{2} \cdot 90 \cdot 72 \cdot \sin 121° \\ &\approx \underline{\hspace{2cm}} \end{aligned}$$



Example 6.1.6. Find the altitude of the aircraft shown in the figure below. Round the altitude to the nearest tenth of a mile.[7]



Solution. Suppose the height of the aircraft is h miles. From the definition of sine function, we have

$$h = a \sin 15°.$$

To find a, we find the angle, denoted as β, with the vertex at the aircraft formed by sides through two radars:

$$\beta = 180° - \underline{\qquad} - \underline{\qquad} = 130°.$$

From the Law of Sines, we have

$$\frac{a}{\sin 35°} = \frac{20 \text{ miles}}{\sin(\underline{\qquad})}$$

$$a = \frac{\underline{\qquad} \cdot 20 \text{ miles}}{\underline{\qquad}} \approx \underline{\qquad} \text{ miles.}$$

Thus, the altitude of the aircraft is

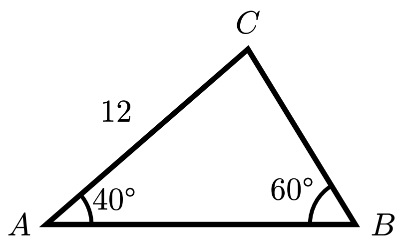
$$h = a \sin 15° \approx \underline{\qquad} \text{ miles.}$$

[7]Source: OpenStax, Precalculus, CC BY-NC-SA 4.0

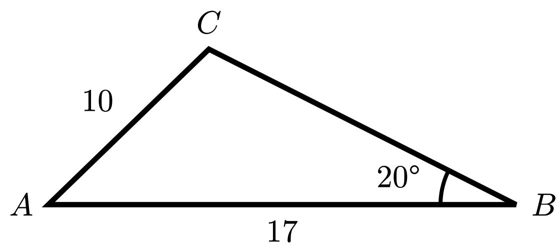
Exercises

Exercise 6.1.1. Solve for the unknown side and angles. Round your answers to the nearest tenth.

1)



2)



Answer: 1) $C \approx 80°$, $AB \approx 13.6$, and $BC \approx 8.9$. 2) $C \approx 11.6°$, $A \approx 148.4°$, and $BC \approx 15.3$.

Exercise 6.1.2. It is known that lengths of two sides of a triangle 15 and 10. The angle opposite to the sides of length 15 is $75\,\mathrm{deg}$.

1) Find the the length of the unknown side.

2) Find area of a triangle.

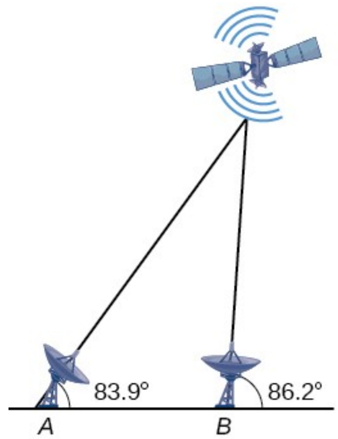
Round your answers to the nearest tenth.

Answer: The length of the third side is approximate 14.1. The area is approximately 67.9.

Exercise 6.1.3. The Figure below shows a satellite orbiting Earth.[8]

The satellite passes directly over two tracking stations A and B, which are 69 miles apart. When the satellite is on one side of the two stations, the angles of elevation at A and B are measured to be $86.2°$ and $83.9°$ respectively.

How far is the satellite from station A and how high is the satellite above the ground? Round answers to the nearest whole mile.



Answer: Satellite is 1716 miles from station A and 1706 miles above the ground.

[8]Source: OpenStax, Precalculus, CC BY-NC-SA 4.0

6.2 Law of Cosines

Theorem 6.2.1 (Law of Cosines)

Given a triangle $\triangle ABC$ with sides of lengths a, b, and c opposite to angles A, B, and C, respectively, then

$$a^2 = b^2 + c^2 - 2bc\cos A \qquad \cos A = \frac{b^2+c^2-a^2}{2bc}$$
$$b^2 = a^2 + c^2 - 2ac\cos B \qquad \text{or} \qquad \cos B = \frac{a^2+c^2-b^2}{2ac}$$
$$c^2 = a^2 + b^2 - 2ab\cos C \qquad \cos C = \frac{a^2+b^2-c^2}{2ab}.$$

Proof. We will only prove the first formula. The other two formulas can be proved similarly.

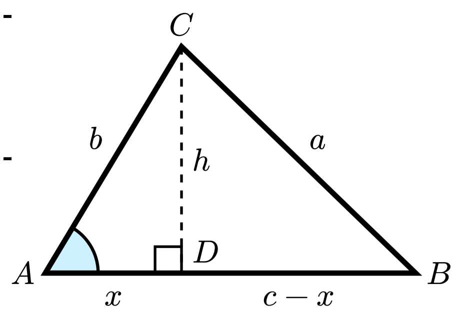
Consider the figure on the right. By the Pythagorean Theorem, we have

$$h^2 + x^2 = b^2 \quad \text{and} \quad h^2 + (c-x)^2 = a^2.$$

Subtracting the first equation from the second and simplifying gives

$$a^2 = b^2 + c^2 - 2cx.$$

From the definition of cosine function, we have



$$x = b\cos A.$$

Substituting this into the equation above gives the desired formula

$$a^2 = b^2 + c^2 - 2bc\cos A.$$

□

Theorem 6.2.2 (Heron's Formulas (Area of Triangle using SSS))

Given a triangle with the sides of lengths a, b, and c, the area is

$$\text{Area} = \sqrt{s(s-a)(s-b)(s-c)},$$

where $s = \frac{a+b+c}{2}$ is the semi-perimeter of the triangle, that is, one half of the perimeter of the triangle.

Proof. Recall that the area can be computed using the formula

$$S = \frac{1}{2}ab\sin C.$$

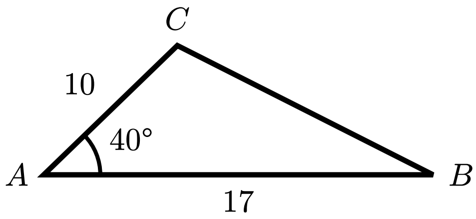
By the Law of Cosines, we have

$$\sin^2 C = 1 - \cos^2 C = 1 - \left(\frac{a^2+b^2-c^2}{2ab}\right)^2.$$

Simplifying $\left(\frac{1}{2}ab\sin C\right)^2$ and comparing with $s(s-a)(s-b)(s-c)$ verifies Herions' formula.

We leave the details to the reader as an exercise. □

Example 6.2.1. Find the length of the unknown side of the triangle.



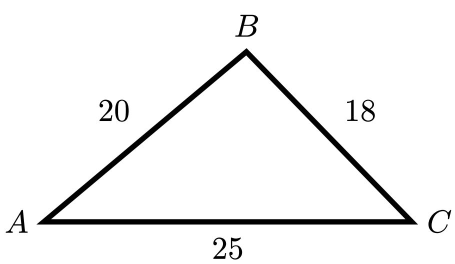
Solution. By the Law of cosine,

$$\begin{aligned} BC^2 &= AC^2 + AB^2 - 2AC \cdot AB \cos 40° \\ &= 17^2 + 10^2 - 2 \cdot 17 \cdot 10 \cdot \cos 40° \\ &= \underline{\hspace{2cm}} - 340 \cos 40°. \end{aligned}$$

Thus,

$$BC = \sqrt{\underline{\hspace{2cm}} - 340 \cos 40°} \approx \underline{\hspace{2cm}}.$$

Example 6.2.2. Find the angles in the triangle. Round your answers to the nearest tenth.



Solution. By the Law of Cosines,

$$\begin{aligned} \cos B &= \frac{25^2 + 20^2 - 18^2}{2 \cdot 25 \cdot 20} \\ &= \underline{\hspace{2cm}}. \end{aligned}$$

Thus,

$$B = \cos^{-1}\left(\underline{\hspace{2cm}}\right) \approx \underline{\hspace{2cm}}.$$

Similarly, we have

$$C = \cos^{-1}\left(\frac{20^2 + 18^2 - 25^2}{2 \cdot 20 \cdot 18}\right) \approx \underline{\hspace{2cm}}.$$

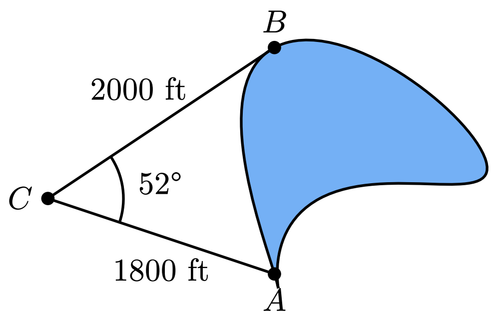
Finally,

$$A = 180° - B - C \approx \underline{\hspace{2cm}}.$$

Rounding Issues

In the above example, A can also be calculated using the Law of Cosines. Due to rounding, the answer may differ slightly from the one obtained by subtracting B and C from $180°$.

Example 6.2.3. To find the distance between two locations A and B across a small lake, a surveyor has taken the measurements shown in the figure below. Find the distance across the lake using this information. Round your answers to the nearest tenth.



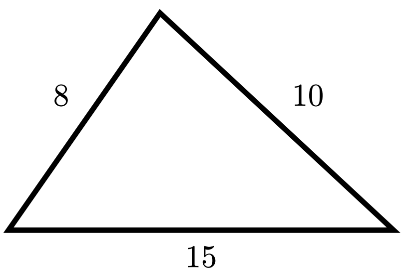
Solution. By the Law of Cosines,

$$\begin{aligned} AB^2 &= 1800^2 + 2000^2 - 2 \cdot 1800 \cdot 2000 \cdot \cos 52^\circ \quad \text{ft}^2 \\ &= \underline{\hspace{3cm}} - \underline{\hspace{3cm}} \cos 52^\circ \quad \text{ft}^2. \end{aligned}$$

Thus,

$$AB = \sqrt{\underline{\hspace{3cm}} - \underline{\hspace{3cm}} \cos 52^\circ} \approx \underline{\hspace{2cm}} \text{ ft.}$$

Example 6.2.4. Find the area of the triangle in the figure below using Heron's formula.



Solution. The semi-perimeter of the triangle is

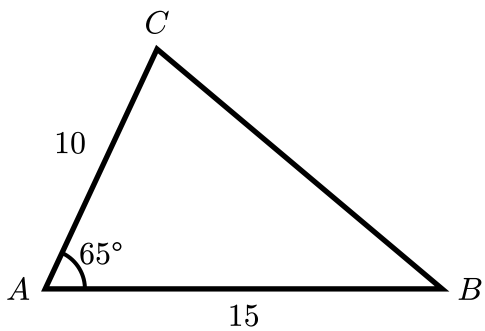
$$s = \frac{15 + 8 + 10}{2} = \underline{\hspace{2cm}}.$$

By Heron's formula, the area of the triangle is

$$\begin{aligned} S &= \sqrt{s(s-15)(s-8)(s-10)} \\ &= \sqrt{\underline{\hspace{2cm}} \cdot \underline{\hspace{2cm}} \cdot \underline{\hspace{2cm}} \cdot \underline{\hspace{2cm}}} \\ &= \sqrt{\underline{\hspace{2cm}}} \\ &\approx \underline{\hspace{2cm}}. \end{aligned}$$

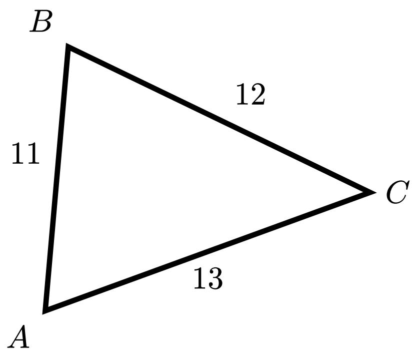
Exercises

Exercise 6.2.1. Find the unknown side and angles of the triangle.



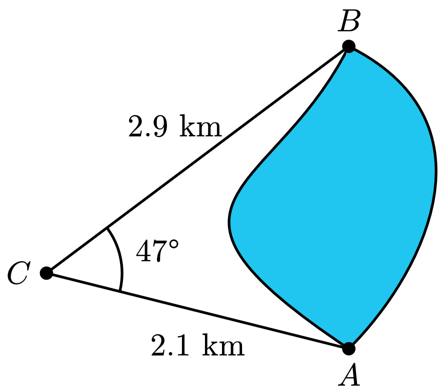
Answer: $BC = 14.08$, $\angle B = 40.1°$, $\angle C = 74.9°$.

Exercise 6.2.2. Find the angles in the triangle. Round your answers to the nearest tenth.



Answer: $A \approx 59.3°$, $B \approx 68.7°$, $C \approx 52°$.

Exercise 6.2.3. To find the distance across a small lake, a surveyor has taken the measurements shown in the figure below. Find the distance across the lake using this information. Round your answers to the nearest tenth.



Answer: The distance across the lake is approximately 2.1 km.

Chapter 7 Conic Sections

7.1 Parabolas

Definition 7.1.1 (Parabolas with Horizontal or Vertical Axis of Symmetry)

A **parabola** is the set of points P in a plane such that the distance from P to a fixed point F (the **focus**) equals its distance to a fixed line l (the **directrix**).

The **axis of symmetry** is the line through the focus, perpendicular to the directrix. The **vertex** is the point where the parabola meets this axis of symmetry. The vertex lies midway between the focus and the directrix and is the point on the parabola that is closest to the directrix.

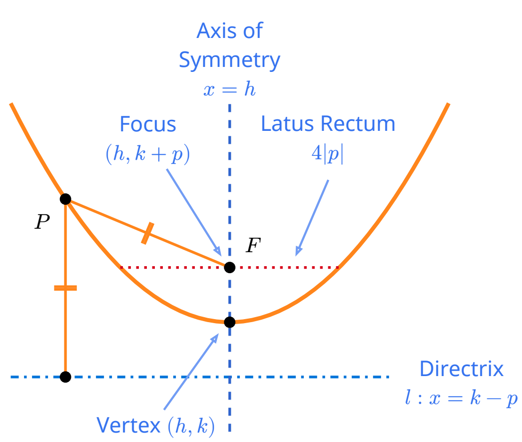
The **latus rectum** is the line segment through the focus, perpendicular to the axis of symmetry, with endpoints on the parabola. Its length is called the **focal diameter**.

Denote by p the *signed distance* along the axis of symmetry from the vertex to the focus (or equivalently to the directrix). Then the focal diameter equals to $|4p|$.

A parabola with a vertical or horizontal axis of symmetry jas the **standard form** equation as follows:

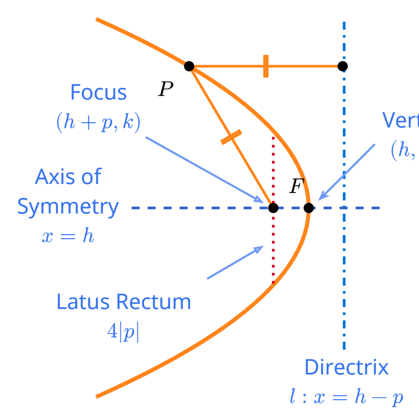
Vertical Axis of Symmetry

$$(x-h)^2 = 4p(y-k)$$



Horizontal Axis of Symmetry

$$4p(x-h) = (y-k)^2$$



If $p > 0$ the parabola opens upward or to the right, and if $p < 0$ it opens downward or to the left.

The focus always lies on the concave side of the parabola while the directrix lies on the opposite side of the parabola. ♣

Conic sections are a broad and fascinating topic. In this chapter, we focus on their basic definitions and standard forms. For more details, see the https://en.wikipedia.org/wiki/Conic_section.

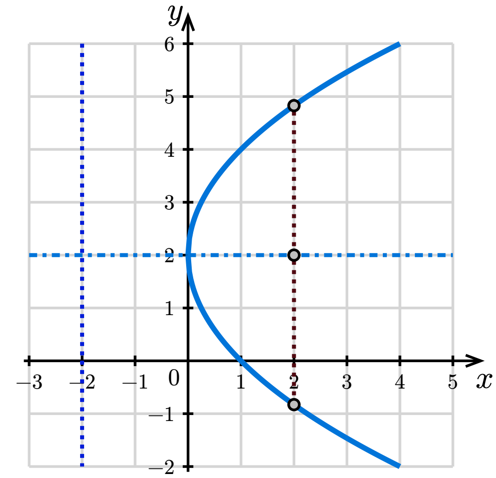
Example 7.1.1. Find an equation of the parabola with the vertex $V(0,1)$ and focus $F(2,1)$, and sketch the graph.

Solution. Since the vertex and the focus have the same y-coordinate, the axis of symmetry is ______. The distance from the vertex to the focus is $2-0=2$, so $p=$ ______.

Using the standard form for a parabola with a horizontal axis of symmetry, we have $4p(x-h)=(y-k)^2$, where $(h,k)=(0,1)$. Thus, an equation of the parabola is

$$16\,\underline{\qquad} = (\underline{\qquad\qquad})^2.$$

To sketch the graph, we plot the vertex at $(0,1)$, the focus at $(2,1)$, the directrix $x=-2$, and the latus rectum with endpoints at $(2,5)$ and $(2,-3)$, then sketch the parabola through the vertex and the endpoints of the latus rectum.



Example 7.1.2. Find the focus, directrix, and focal diameter of the parabola $y=\frac{1}{2}x^2$.

Solution. Rewriting the equation in standard form, we have

$$x^2 = 4\left(\underline{\qquad}\right)y.$$

Thus, the vertex is at ______ and $p=$ ______.

Therefore, the focus is at ______, the directrix is the line $y=$ ______. The focal diameter is double the distance between the focus and the directrix and quadruple the distance from the vertex to the focus or the directrix, that is

$$|4p| = \frac{1}{2}.$$

Example 7.1.3. Find an equation of the parabola with the focus $(1,2)$ and the directrix $y=-2$.

Solution. The vertex is the midpoint between the focus and the directrix, which is at

$$(1, \underline{\qquad}).$$

The distance from the vertex to the focus is $2-0=2$, so

$$p = \underline{\qquad}.$$

Using the standard form for a parabola with a vertical axis of symmetry, we have

$$(x-h)^2 = 4p(y-k),$$

where $(h,k)=(1,0)$. Thus, an equation of the parabola is

$$(\underline{\qquad\qquad})^2 = 8\,\underline{\qquad}.$$

Example 7.1.4. A searchlight has a parabolic reflector that forms a “bowl,” which is 12 in. wide from rim to rim and 8 in. deep. If the filament of the light bulb is located at the focus, how far is the focus from the bottom of the reflector?

Solution. Let the vertex of the parabola be at the origin $(0, 0)$, and let the y-axis be the axis of symmetry. Since the reflector is 12 in. wide from rim to rim, and 8 in. deep, the points $(6, 8)$ lie on the parabola. Because a parabola with a vertical axis of symmetry and with vertex at the origin, the standard form equation of the parabola is

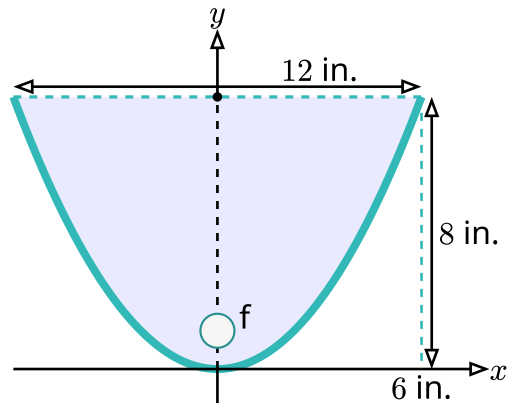
$$x^2 = 4py.$$

Substituting the point $(6, -8)$ into the equation, we have

$$36 = 4p(-8)$$

$$p = \underline{\qquad\qquad}.$$

Thus, the focus is at $(0, \underline{\qquad\qquad})$. Therefore, the focus is $\underline{\qquad\qquad}$ in. from the bottom of the reflector.



Example 7.1.5. Find the vertex, focus, and directrix for the following parabola $3x - 5 = y^2 - 4y$.

Solution. Rewriting the equation in standard form by completing the square for y, we have

$$y^2 - 4y + 4 = 3x - 5 + \underline{\qquad\qquad}$$

$$(y - 2)^2 = 4\left(\underline{\qquad\quad}\right)(x - \underline{\qquad\qquad}).$$

Thus, the vertex is at $\underline{\qquad\qquad\qquad}$ and $p = \underline{\qquad\qquad}$.

Therefore, the focus is at $\underline{\qquad\qquad\qquad}$, and the directrix is the line $x = \underline{\qquad\qquad}$.

Exercises

Exercise 7.1.1. Find the vertex, focus, and directrix of the parabola.

1) $x^2 = -8(y-1)$. 2) $(y+1)^2 = 12(x-2)$. 3) $x^2 + 2x + 4y = 3$.

Answer: 1) Vertex: $(0,1)$; Focus: $(0,-1)$; Directrix: $y=3$. 2) Vertex: $(2,-1)$; Focus: $(5,-1)$; Directrix: $x=-1$. 3) Vertex: $(-1,-1)$; Focus: $(-1,0)$; Directrix: $y=2$.

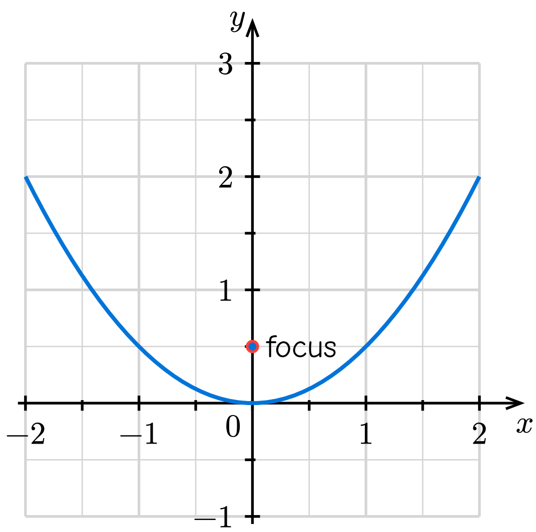
Exercise 7.1.2. Find an equation for the conic section with the given properties.

1) The parabola with vertex at $(1,0)$ and focus $(1,5)$.

2) The parabola with vertex at $(2,1)$ and the directrix $x=-2$.

Answer: 1) $(x-1)^2 = 20(y-0)$. 2) $(y-1)^2 = 8(x-2)$.

Exercise 7.1.3. Find the standard form equation for the parabola whose graph is given below.



Answer: $x^2 = 2y$.

7.2 Ellipses

Definition 7.2.1 (Ellipses)

An **ellipse** is the set of points P in the plane such that the sum of distances from P to two fixed points F_1 and F_2, called the **foci** (plural of focus), is a constant $2a$. The midpoint between the foci is called the **center** of the ellipse. The distance from the center to each focus is denoted by c.

The **major axis** is the longest diameter of the ellipse, and the **minor axis** is the shortest diameter of the ellipse. The major axis passes through both foci, while the minor axis is perpendicular to the major axis at the center. The lengths of the major axis is $2a$ and the minor axis is $2b$ where $b = \sqrt{a^2 - c^2}$, or equivalently

$$a^2 = b^2 + c^2.$$

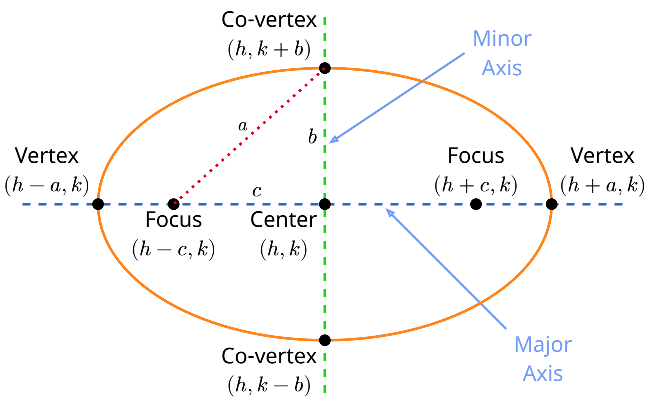
The intersections of the ellipse and the major axis are called the **vertices**, and the intersections of the ellipse and the minor axis are called the **co-vertices**.

The center is also the midpoint of the vertices, or the co-vertices.

An ellipse with horizontal or vertical major axis has a **standard form** equation as follows.

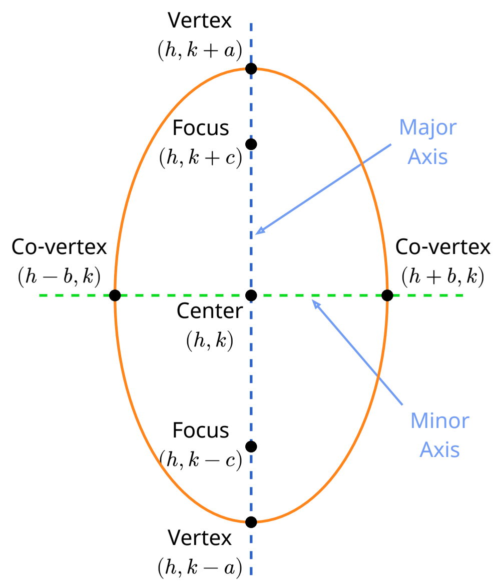
Horizontal Major Axis

$$\frac{(x-h)^2}{a^2} + \frac{(y-k)^2}{b^2} = 1$$



Vertical Major Axis

$$\frac{(y-k)^2}{a^2} + \frac{(x-h)^2}{b^2} = 1$$



The distance c from each focus to the center is called the **linear eccentricity**.

The eccentricity e of an ellipse is

$$e = \frac{\text{linear eccentricity}}{\text{semi major axis}} = \frac{c}{a},$$

which shows how much the ellipse differs from a circle, with $0 \le e < 1$. The closer e is to 0, the more the ellipse looks like a circle. ♣

Example 7.2.1. An ellipse has the equation $\frac{x^2}{9} + \frac{y^2}{4} = 1$. Find the foci, the vertices, and the lengths of the major and minor axes. Sketch the graph.

Solution. The given equation is in standard form with

$$h = 0, \quad k = 0, \quad a^2 = 9, \quad \text{and} \quad b^2 = 4.$$

Thus, the center is at $(0, 0)$, the vertices are at $(\pm 3, 0)$, and the co-vertices are at $(0, \pm 2)$.

The length of the major axis is

$$2a = \underline{\hspace{2cm}},$$

and the length of the minor axis is

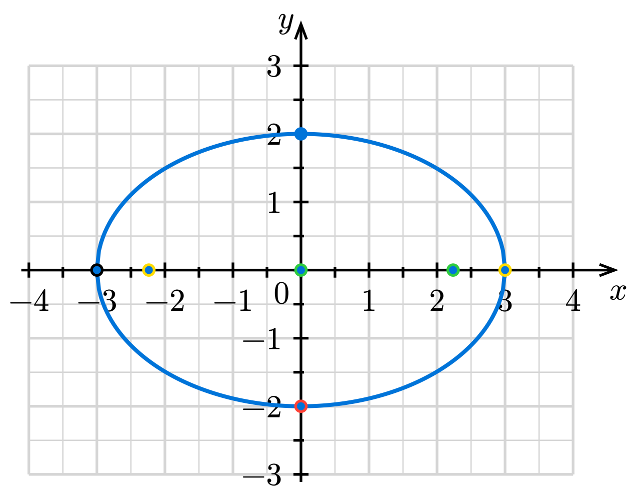
$$2b = \underline{\hspace{2cm}}.$$

To find the foci, we use the relationship $a^2 = b^2 + c^2$ to find c:

$$9 = 4 + c^2$$
$$c^2 = \underline{\hspace{2cm}}$$
$$c = \underline{\hspace{2cm}}.$$

Therefore, the foci are at $(\pm \underline{\hspace{1.5cm}}, 0)$.

To sketch the graph, we plot the center, vertices, and co-vertices, and then draw a smooth curve through these points to form the ellipse.



Latus Rectum and Directrix of an Ellipse

The **latus rectum** of an ellipse is a line segment perpendicular to the major axis that passes through a focus and has endpoints on the ellipse. The length is $\frac{2b^2}{a}$.

For an ellipse centered at (h, k), the endpoints through the foci are

- $\left(h \pm c, k \pm \left(\frac{b^2}{a}\right)\right)$ if the major axis is horizontal, and
- $\left(h \pm \left(\frac{b^2}{a}\right), k \pm c\right)$ if the major axis is vertical.

Plotting these endpoints along with the center, vertices, and foci gives a more accurate sketch of the ellipse.

To have a more accurate sketch of an ellipse, we can plot the endpoints of the latus rectum in addition to the center, vertices, and foci.

An ellipse can also be defined as the set of points where the ratio of the distance to a focus and the distance to its corresponding directrix is constant—the eccentricity e. The directrices are two lines perpendicular to the major axis, located at a distance $\frac{a^2}{c}$ from the center.

Example 7.2.2. Find the foci of the ellipse $16x^2 + 9(y-2)^2 = 144$.

Solution. Rewriting the equation in standard form, we have

$$\frac{(x-0)^2}{9} + \frac{(y-2)^2}{16} = 1.$$

Thus, the center is at $(0, 2)$,

$$a^2 = \underline{\hspace{2cm}}, \quad \text{and} \quad b^2 = 9.$$

To find the foci, we use the relationship $a^2 = b^2 + c^2$ to find c:

$$16 = 9 + c^2$$
$$c^2 = \underline{\hspace{2cm}}$$
$$c = \underline{\hspace{2cm}}.$$

Since $a^2 > b^2$, the major axis is vertical. The foci have the same x-coordinate as the center, and their y-coordinates are found by adding and subtracting c from the y-coordinate of the center. Therefore, the foci are at

$$(\underline{\hspace{1.5cm}}, 2 \pm \underline{\hspace{1.5cm}}).$$

Example 7.2.3. Find an equation of the ellipse with the vertices $(\pm 4, 1)$ and the foci $(\pm 2, 1)$.

Solution. Since the vertices and foci are on the same vertical line $y = 1$, the major axis of the ellipse is vertical and hence the equation in the standard form is

$$\frac{(x-h)^2}{b^2} + \frac{(y-k)^2}{a^2} = 1.$$

Here (h, k) is the center which is also the midpoint of the vertices or foci and given by

$$(h, k) = \left(\frac{4 + (-4)}{2}, \frac{1+1}{2}\right) = (0, 1).$$

From the vertices, we have the length of the semi-major axis, which is the half distance between vertices, is

$$a = \underline{\hspace{2cm}}.$$

From the foci, we have the distance from the center to each focus, which is the linear eccentricity, is

$$c = \underline{\hspace{2cm}}.$$

Using the relationship $a^2 = b^2 + c^2$, we can find b^2:

$$16 = b^2 + 4$$
$$b^2 = \underline{\hspace{2cm}}.$$

Thus, an equation of the ellipse is

$$\frac{x^2}{\underline{\hspace{1.5cm}}} + \frac{(\underline{\hspace{2cm}})^2}{16} = 1.$$

Example 7.2.4. Find the equation of the ellipse with foci $(0, \pm 8)$ and the eccentricity $e = \frac{4}{5}$.

Solution. Since the foci have the same x-coordinate, the major axis of the ellipse is vertical and hence the equation in the standard form is

$$\frac{(x-h)^2}{b^2} + \frac{(y-k)^2}{a^2} = 1.$$

Here (h, k) is the center which is also the midpoint of the foci and given by

$$(h, k) = \left(\frac{0+0}{2}, \frac{8+(-8)}{2}\right) = (0, 0).$$

From the foci, we have the distance from the center to each focus, which is the linear eccentricity, is $c =$ ________.

Using the eccentricity, we can find a:

$$e = \frac{c}{a}$$

$$\frac{4}{5} = \frac{8}{a}$$

$$a = \underline{\qquad\qquad}.$$

Using the relationship $a^2 = b^2 + c^2$, we can find b^2:

$$64 = b^2 + \frac{64}{25}$$

$$b^2 = \underline{\qquad\qquad}.$$

Thus, an equation of the ellipse is $\dfrac{x^2}{\underline{\qquad}} + \dfrac{y^2}{\underline{\qquad}} = 1.$

Exercises

Exercise 7.2.1. An equation of an ellipse is given. Find the center, vertices, and foci of the ellipse, and the lengths of the major and minor axes.

1) $\frac{x^2}{9} + \frac{y^2}{25} = 1.$
2) $\frac{(x-1)^2}{25} + \frac{(y+1)^2}{9} = 1.$
3) $9x^2 + 18x + 25y^2 = -8.$

Answer:
1) Center: $(0, 0)$; Vertices: $(0, \pm 5)$; Foci: $(0, \pm 4)$; Major axis length: 10; Minor axis length: 6.
2) Center: $(1, -1)$; Vertices: $(6, -1)$, (4, −1); Foci: $(5, -1)$, $(-3, -1)$; Major axis length: 10; Minor axis length: 6.
3) Center: $(-1, 0)$; Vertices: $(-\frac{2}{3}, 0)$, $(-\frac{4}{3}, 0)$; Foci: $(-\frac{11}{15}, 0)$, $(-\frac{19}{15}, 0)$; Major axis length: $\frac{2}{3}$; Minor axis length: $\frac{2}{5}$.

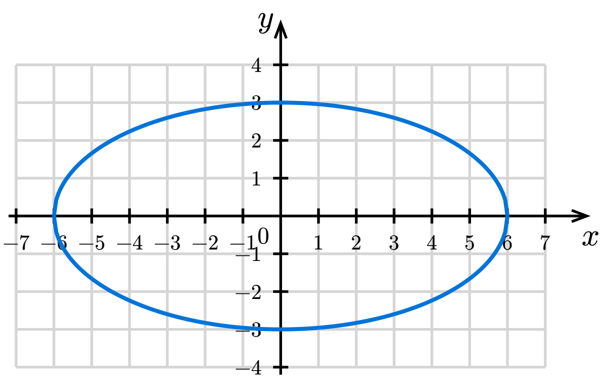
Exercise 7.2.2. Find an equation for the ellipse with each set of given properties.

1) vertices $(\pm 2, 0)$ and foci $(\pm 1, 0)$.

2) foci $(1, 4)$ and $(1, 0)$, and the eccentricity $e = \frac{4}{5}$.

Answer: 1) $\frac{x^2}{4} + \frac{y^2}{3} = 1$. 2) $\frac{(x-1)^2}{25} + \frac{(y-2)^2}{9} = 1$.

Exercise 7.2.3. Find an question for the ellipse with the given graph.



Answer: $\frac{x^2}{36} + \frac{y^2}{9} = 1$.

7.3 Hyperbola

Definition 7.3.1 (Hyperbolas)

A **hyperbola** is the set of points P in the plane such that the absolute difference of the distances from P to two fixed points F_1 and F_2, called the **foci**, is a constant $2a$.

The midpoint of foci is the **center** of the hyperbola. The distance from each focus to the center is denoted by c.

A hyperbola has two separate curves called **branches**. Each branch approaches two lines through the center called **asymptotes**.

The **transverse axis** is the shortest line segment connecting the two branches of the hyperbola. The endpoints of the transverse axis are called the **vertices** of the hyperbola. The transverse axis passes through the foci and has the length $2a$.

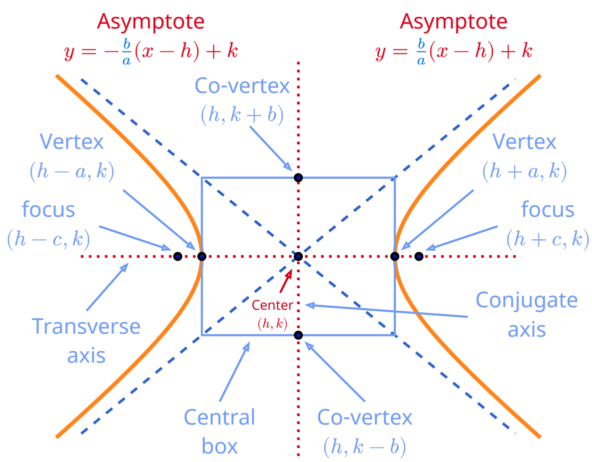
The rectangle whose diagonals lie along the asymptotes and with a side passing through a vertex is called the **central box**.

The line segment through the center, perpendicular to the transverse axis, with end-points on the central box is the **conjugate axis**. Its endpoints are the **co-vertices**.

The **standard form** of a hyperbola with a horizontal or vertical transverse axis is one of the following:

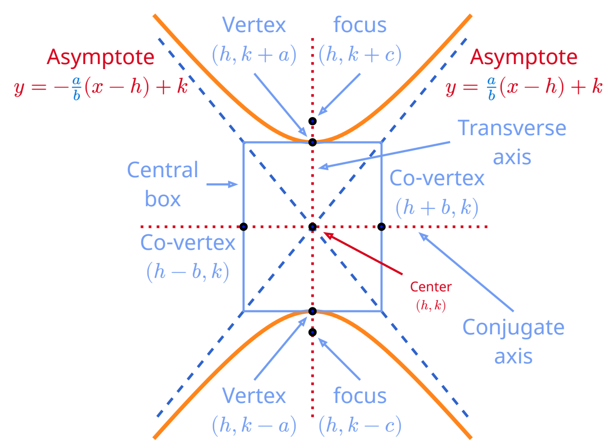
Horizontal Transverse Axis

$$\frac{(x-h)^2}{a^2} - \frac{(y-k)^2}{b^2} = 1$$



Vertical Transverse Axis

$$\frac{(y-k)^2}{a^2} - \frac{(x-h)^2}{b^2} = 1$$



In the equations of standard form and figures above, b is defined by $b = \sqrt{c^2 - a^2}$. ♣

Equations of Asymptotes

In standard form equation of a hyperbola, replacing 1 with 0 and solving for y by factoring yields the equations of the asymptotes. Conversely, the product of the equations of the asymptotes is differ by a constant with an equation of the hyperbola.

Example 7.3.1. A hyperbola has the equation $9x^2 - 16y^2 = 121$. Find the vertices, foci, length of the transverse axis, and asymptotes. Sketch the graph.

Solution. Rewriting the equation in standard form, we have

$$\frac{x^2}{\frac{121}{9}} - \frac{y^2}{\frac{121}{16}} = 1.$$

Thus, the center is at $(0, 0)$,

$$a = \underline{\hspace{2cm}}, \quad \text{and} \quad b = \underline{\hspace{2cm}}.$$

The vertices are located a units from the center along the transverse axis. Since the transverse axis is horizontal, the vertices are at

$$(\pm \underline{\hspace{2cm}}, 0).$$

To find the foci, we use the relationship $c^2 = a^2 + b^2$ to find c:

$$c = \sqrt{\frac{121}{9} + \frac{121}{16}} = \underline{\hspace{2cm}}.$$

Since the transverse axis is horizontal, the foci are at

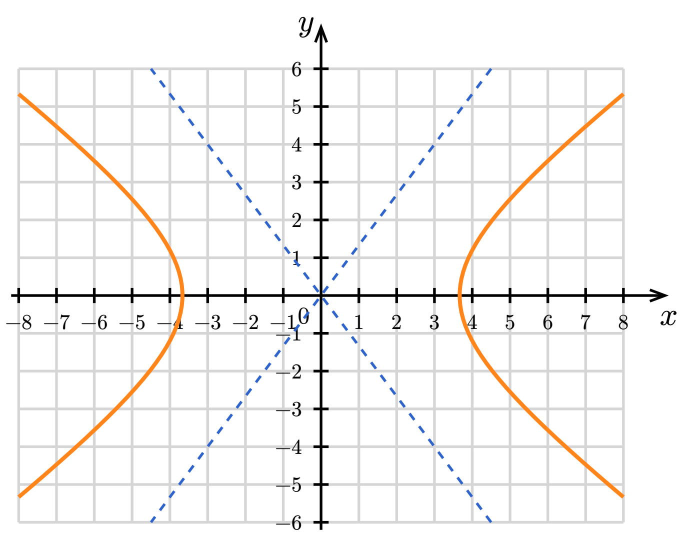
$$(\pm \underline{\hspace{2cm}}, 0).$$

The length of the transverse axis is $2a = \underline{\hspace{2cm}}$.

The equations of the asymptotes are given by

$$y = \pm\left(\frac{b}{a}\right)x = \pm\left(\frac{\underline{\hspace{2cm}}}{\underline{\hspace{2cm}}}\right)x.$$

To sketch the graph, we first plot the center, vertices, and foci. Then we draw the asymptotes as dashed lines through the center. Finally, we sketch the two branches of the hyperbola, approaching but never touching the asymptotes.



Latus Rectum and Directrix of a Hyperbola

The latus rectum of a hyperbola is a line segment perpendicular to the transverse axis that passes through a focus and has endpoints on the hyperbola. The length of the latus rectum is $\frac{2b^2}{a}$.

For a hyperbola centered at (h, k), the endpoints of the latus rectum through the foci are:

- $\left(h \pm c, k \pm \left(\frac{b^2}{a}\right)\right)$ if the transverse axis is horizontal.
- $\left(h \pm \left(\frac{b^2}{a}\right), k \pm c\right)$ if the transverse axis is vertical.

Plotting these endpoints along with the center, vertices, and foci gives a more accurate sketch of the hyperbola.

An ellipse can also be defined as the set of points where the ratio of the distance to a focus and the distance to its corresponding directrix is constant, the eccentricity e. The directrices of an ellipse are the two lines perpendicular to the transverse axis and located a distance of $\frac{a^2}{c}$ from the center.

Example 7.3.2. Find the vertices, foci, length of the transverse axis, and asymptotes of the hyperbola $x^2 + 2x - 9y^2 + 10 = 0$.

Solution. Completing the square for x and rewriting the equation in standard form gives

$$(x^2 + 2x + 1) - 9y^2 = \underline{\qquad\quad}$$

$$y^2 - \frac{(x+1)^2}{\underline{\qquad\quad}} = 1.$$

Thus, the center is at $(-1, 0)$,

$$a = 1, \quad \text{and} \quad b = \sqrt{\underline{\qquad\quad}} = \underline{\qquad\quad}.$$

The vertices are located a units from the center along the transverse axis. Since the transverse axis is horizontal, the vertices are at

$$(-1 + \underline{\qquad\quad}, 0) = \underline{\qquad\quad}, \quad \text{and} \quad (-1 - \underline{\qquad\quad}, 0) = \underline{\qquad\quad}.$$

From the equation $c^2 = a^2 + b^2$, we find

$$c = \sqrt{10 + \frac{10}{9}} = \underline{\qquad\quad}.$$

Since the transverse axis is horizontal, the foci are at

$$(-1 + \underline{\qquad\quad}, 0) = \underline{\qquad\quad}, \quad \text{and} \quad (-1 - \underline{\qquad\quad}, 0) = \underline{\qquad\quad}.$$

The length of the transverse axis is $2a = \underline{\qquad\quad}$

The equations of the asymptotes are given by

$$y = \pm\left(\frac{b}{a}\right)(x - h) + k = \pm\left(\frac{\underline{\qquad\quad}}{\underline{\qquad\quad}}\right)(x + 1).$$

Example 7.3.3. Find the equation of the hyperbola with vertices $(\pm 3, 1)$ and foci $(\pm 4, 1)$.

Solution. The center is the midpoint of the foci:

$$\frac{(3,1)+(-3,1)}{2} = (0, \underline{\hspace{2em}}).$$

The distance from the center to a vertex is

$$a = \underline{\hspace{2em}}.$$

The distance from the center to a focus is

$$c = \underline{\hspace{2em}}.$$

Using the relationship $c^2 = a^2 + b^2$, we find b^2:

$$b^2 = c^2 - a^2 = 16 - 9 = \underline{\hspace{2em}}.$$

Since the foci and vertices are on the same horizontal lines $y = 1$, the transverse axis is horizontal, the equation of the hyperbola is

$$\frac{x^2}{\underline{\hspace{2em}}} - \frac{(y-1)^2}{9} = 1.$$

Example 7.3.4. Find an equation of the hyperbola with vertices $(\pm 2, 1)$ and asymptotes $y = \pm\frac{1}{2}x + 1$.

Solution. (Using the center, a, and b). The center is the midpoint of the vertices: $(0, 1)$. The distance from the center to a vertex is $a = \underline{\hspace{2em}}$.

From the slope of the equations of the asymptotes, we have

$$\frac{a}{c} = \frac{1}{2}$$
$$c = \underline{\hspace{2em}}.$$

Using the relationship $b^2 = c^2 - a^2$, we find b^2:

$$b^2 = \left(\frac{a}{2}\right)^2 = \frac{4}{4} = \underline{\hspace{2em}}.$$

Since the vertices are on the horizontal line $y = 1$, the transverse axis is horizontal. Therefore, the equation of the hyperbola is

$$\frac{x^2}{4} - \frac{(y-1)^2}{\underline{\hspace{2em}}} = 1.$$

Solution. (Using equations of asymptotes). Since the asymptotes are given by $(y-1) + \pm\frac{1}{2}x = 0$, an equation of the hyperbola is of the form

$$\left((y-1) + \frac{1}{2}x\right)\left((y-1) - \frac{1}{2}x\right) = k$$

for some constant $k\neg = 0$. Since (2, 1) is a vertex, pluging this point into the equation gives

$$k = \left(1 - 1 + \frac{1}{2} \cdot 2\right)\left(1 - 1 - \frac{1}{2} \cdot 2\right) = \underline{\hspace{2em}}.$$

Thus, an equation of the hyperbola is

$$\frac{x^2}{4} - (y-1)^2 = 1.$$

Exercises

Exercise 7.3.1. An equation of a hyperbola is given. Find the center, vertices, foci, and asymptotes of the hyperbola. Sketch the graph.

1) $\frac{x^2}{9} - \frac{y^2}{25} = 1.$

2) $\frac{y^2}{9} - \frac{x^2}{25} = 1.$

3) $25x^2 - 9y^2 - 4 = 0.$

Answer:

1) Center: $(0,0)$; Vertices: $(\pm 3, 0)$; Foci: $\left(\pm\sqrt{34}, 0\right)$; Asymptotes: $y = \pm\frac{5}{3}x$.

2) Center: $(0,0)$; Vertices: $(0, \pm 3)$; Foci: $\left(0, \pm\sqrt{34}\right)$; Asymptotes: $y = \pm\frac{3}{5}x$.

3) Center: $(0,0)$; Vertices: $(\pm\frac{2}{5}, 0)$; Foci: $\left(\pm\frac{2\sqrt{34}}{15}, 0\right)$; Asymptotes: $y = \pm\frac{3}{5}x$.

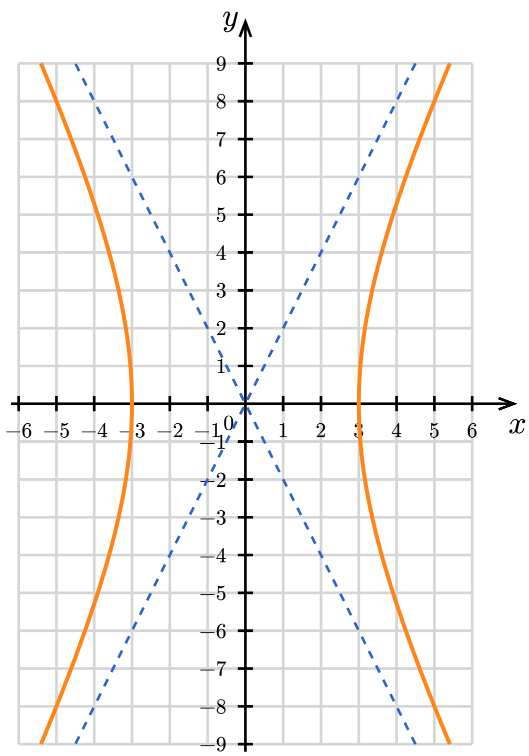
Exercise 7.3.2. Find an equation for the conic section with the given properties.

1) The hyperbola with foci $(0, \pm 3)$ and vertices $(\pm 2, 0)$.

2) The hyperbola with foci $(\pm 5, 1)$ and asymptotes $y = \pm\frac{3}{4} + 1$.

Answer: 1) $\frac{x^2}{4} - \frac{y^2}{5} = 1$. 2) $\frac{x^2}{\frac{16}{25}} - \frac{(y-1)^2}{\frac{9}{25}} = 1$.

Exercise 7.3.3. Find an question for the conic section with the given graph.



Answer: $\frac{x^2}{9} - \frac{y^2}{36} = 1.$

Chapter 8 Sequences and Series

8.1 Sequences

Definition 8.1.1 (Sequences)

A **sequence** is an ordered list of numbers or equivalently a function whose domain is the set of positive integers or right tail truncated set of integers. Each number in the sequence is called a **term**. Sequences can be finite or infinite. A sequence is often denoted by $\{a_n\}$, where a_n is called the **n-th term** or **general term** of the sequence and n is called the **index** of the sequence. ♣

ⓘ Sequences as Functions

A sequence can be viewed as a function whose domain is the set of positive integers $\mathbb{Z}^+ = \{1, 2, 3, \cdots\}$ or a truncated or extended set of integers $\{m, m+1, m+2, \cdots\}$ for some integer m. Thus, a sequence $\{a_n\}$ can be defined by a formula for its n-th term $a_n = f(n)$ for some function f defined on $\mathbb{Z}^+$ or $\{m, m+1, m+2, \cdots\}$.

Example 8.1.1. Find the first five terms and the 100-th term of the sequence defined by each formula.

1) $a_n = 2n^2 - 1$

2) $r_n = \frac{(-1)^n}{2^n}$

Solution. To find a term in a sequence defined by a formula, we substitute the index of the term into the formula.

1) For $a_n = 2n^2 - 1$, the first five terms are $a_1 = 1$, $a_2 = 7$, $a_3 = 17$, $a_4 = 31$, $a_5 = 49$, and the 100-th term is $a_{100} = 2 \cdot 100^2 - 1 =$ ______.
2) For $r_n = \frac{(-1)^n}{2^n}$, the first five terms are $r_1 = -\frac{1}{2}$, $r_2 = \frac{1}{4}$, $r_3 = -\frac{1}{8}$, $r_4 = \frac{1}{16}$, $r_5 = -\frac{1}{32}$, and the 100-th term is $r_{100} = \frac{1}{2^{100}}$.

Example 8.1.2. Find the n-th term of a sequence whose first several terms are given. The ellipsis $\cdots$ indicates that the pattern continues.

1) $\frac{1}{2}, \frac{3}{4}, \frac{4}{5}, \frac{5}{6}, \cdots$

2) $-2, 4, -8, 16, \cdots$

Solution. To find the n-th term of a sequence from its first few terms, look for a pattern often by examining differences or ratios between terms.

1) Observe that the numerator of each term increases by 1 starting from 1, and the denominator also increases by 1 starting from 2. Thus, the n-th term is $a_n = \frac{n}{n+1}$.
2) Observe that the absolute value of each term is a power of 2, and the sign alternates. Thus, the n-th term is $a_n = (-1)^n \cdot 2^n$.

Definition 8.1.2 (Recursive Sequences)

In some sequences, the n-th term may depend on some or all of the terms preceding it. Such a sequence is called **recursive sequence**.

Example 8.1.3. Find the first five terms of the sequence defined recursively by $a_1 = 1$ and $a_n = 3(a_{n-1} + 2)$.

Solution. To find the first five terms of the sequence, we use the recursive formula step by step.

$$a_1 = 1, \quad a_2 = 3(a_1 + 2) = 3(\underline{\qquad} + 2) = \underline{\qquad}, \quad a_3 = 3(a_2 + 2) = 3(9 + 2) = \underline{\qquad},$$
$$a_4 = 3(a_3 + 2) = 3(\underline{\qquad} + 2) = \underline{\qquad}, \quad a_5 = 3(a_4 + 2) = 3(105 + 2) = 321$$

Thus, the first five terms are 1, 9, 33, 105, and 321.

Example 8.1.4. Find the first seven terms of the **Fibonacci sequence** defined recursively by $F_1 = 1$, $F_2 = 1$ and

$$F_n = F_{n-1} + F_{n-2}.$$

Solution. To equation for the n-th term shows that the n-term is the sum of previous two terms in the Fibonacci sequence. Thus, we have

$$F_1 = 1, \quad F_2 = 1, \quad F_3 = F_2 + F_1 = 1 + 1 = 2, \quad F_4 = F_3 + F_2 = 2 + 1 = 3,$$
$$F_5 = F_4 + F_3 = \underline{\qquad} + \underline{\qquad} = \underline{\qquad},$$
$$F_6 = F_5 + F_4 = \underline{\qquad} + \underline{\qquad} = \underline{\qquad},$$
$$F_7 = F_6 + F_5 = \underline{\qquad} + \underline{\qquad} = \underline{\qquad}.$$

Therefore, the first seven terms of the Fibonacci sequence are 1, 1, 2, 3, 5, 8, and 13.

Definition 8.1.3 (Partial Sums)

For the sequence $\{a_n\}$, the sum $S_n = a_1 + a_2 + \cdots + a_n$ for first n terms is called the ***n*-th partial sum**. The sum of the entire sequence is called the **sum** of the sequence.

Example 8.1.5. Consider the sequence $\{a_n\}$ defined by $a_n = \frac{1}{n} - \frac{1}{n+1}$. Find the partial sums S_3 and the n-th partial sum S_n.

Solution. To find the partial sums, we evaluate the sum of the first n-terms

$$S_3 = \left(\frac{1}{1} - \frac{1}{2}\right) + \left(\frac{1}{2} - \frac{1}{3}\right) + \left(\frac{1}{3} - \frac{1}{4}\right)$$
$$= 1 + \left(\left(-\frac{1}{2}\right) + \frac{1}{2}\right) + \left(\left(-\frac{1}{3}\right) + \frac{1}{3}\right) - \frac{1}{4} = 1 - \frac{1}{4} = \underline{\qquad},$$

Note that the terms cancel out in pairs except the first term 1 and the last term $-\frac{1}{n+1}$. Thus, the n-th partial sum is

$$S_n = a_1 + a_2 + \cdots + a_n = 1 - \frac{1}{n+1} = \frac{n}{n+1}.$$

Definition 8.1.4 (Sigma Notation)

Given a sequence $\{a_n\}$ and integers $p < q$, the sum $a_p + a_{p+1} + \cdots + a_q$ is often denoted by the summation notation as follows:

$$\sum_{k=p}^{q} a_k$$

where the Greek letter **sigma** $\sum$ **means to sum expressions up**, k is the **index of summation**, and q is the **upper limit of summation**, p is **the lower limit of summation**, and the $\boldsymbol{k}$**-th summand** $\boldsymbol{a_k}$ is the k-th term of the sequence.

Example 8.1.6. Find the sum.

1) $\sum_{k=1}^{5} k^2$

2) $\sum_{j=3}^{5} \frac{1}{j}$

Solution. From the definedion of summation notation, we have

$$\sum_{k=1}^{5} k^2 = 1^2 + 2^2 + 3^2 + 4^2 + 5^2 = \underline{\qquad}$$

and

$$\sum_{j=3}^{5} \frac{1}{j} = \frac{1}{3} + \frac{1}{4} + \frac{1}{5} = \underline{\qquad}.$$

Example 8.1.7. Write each sum using sigma notation.

1) $1^3 + 2^3 + 4^3 + \cdots + 7^3$

2) $\sqrt{1} + \sqrt{3} + \sqrt{5} + \cdots + \sqrt{13}$

Solution. To write the sums using sigma notation, we identify the pattern of the terms.

1) The terms are cubes of integers from 1 to 7. Thus, the sum can be written as

$$\sum_{k=\underline{\qquad}}^{\underline{\qquad}} k^3.$$

2) The terms are square roots of odd integers from 1 to 13. Thus, the sum can be written as

$$\sum_{j=\underline{\qquad}}^{\underline{\qquad}} \sqrt{2j-1}.$$

Proposition 8.1.5 (Properties of Partial Sums)

Let $\{a_n\}$ and $\{b_n\}$ be two sequences.

1) $\sum_{k=1}^{n} (c \cdot a_k + d \cdot b_k) = c \sum_{k=1}^{n} a_k + d \sum_{k=1}^{n} b_k$ for any constants c and d.

2) $\sum_{k=1}^{n} a_k = \sum_{k=1}^{m} a_k + \sum_{k=m+1}^{n} a_k$ for any $1 < m < n$.

Proof. The proofs follow directly from the definitions of summation notation and partial sums, and rules of arithmetic. □

Exercises

Exercise 8.1.1. Find the first 5 terms of the sequence with the given n-th term.

1) $a_n = \frac{n^2}{n+1}$

2) $a_n = (-1)^n \frac{2^n}{n}$

Answer: 1) $\frac{1}{2}, \frac{4}{3}, \frac{9}{4}, \frac{16}{5}, \frac{25}{6}$. 2) $-2, 2, -\frac{8}{3}, 4, -\frac{32}{5}$.

Exercise 8.1.2. Find the first 5 terms of the recursive sequence.

1) $a_n = a_{n-1} + 2n - 1$, $a_1 = 1$

2) $a_n = a_{n-1} - a_{n-2}$, $a_1 = 1$ and $a_2 = 2$

Answer: 1) $1, 4, 9, 16$, and 25. 2) $1, 2, -1, 3$, and -4.

Exercise 8.1.3. Find the partial sum S_4 for the sequence.

1) $\{k^3\}$

2) $\left\{\frac{1}{j+1}\right\}$

Answer: 1) $S_4 = 100$. 2) $S_4 = \frac{77}{60}$.

Exercise 8.1.4. Write each sum using sigma notation.

1) $1^3 + 3^3 + 5^3 + \cdots + 11^3$

2) $\sqrt[3]{1} + \sqrt[3]{2} + \sqrt[3]{3} + \cdots + \sqrt[3]{10}$

Answer: 1) $\sum_{k=0}^{5} (2k+1)^3$. 2) $\sum_{j=1}^{10} \sqrt[3]{j}$.

Exercise 8.1.5. Find the sum.

1) $\sum_{k=1}^{10} (k-1)^2$

2) $\sum_{i=2}^{7} \frac{2i}{2i-1}$

3) $\sum_{j=1}^{3} \frac{(-2)^j}{j+1}$

Answer: 1) 285. 2) $\frac{1229}{105}$. 3) $-\frac{14}{3}$.

8.2 Arithmetic Sequences

Definition 8.2.1 (Arithmetic Sequences)

An **arithmetic sequence** is a linear function whose n-th term is $a_n = d(n-m) + a_m$, where the slope d is called the **common difference**.

If we denote the first a_1 as a, then the n-th term of an arithmetic sequence is given by $a_n = a + (n-1)d$, where $d = a_{k+1} - a_k$ for any positive integer k.

Example 8.2.1. Find a_n for the arithmetic sequence

$$9, 4, -1, -6, -11, \cdots$$

Solution. Because the sequence is arithmetic, its a_n term can be written as $a_n = a_1 + d(n-1)$, where $d = a_{k+1} - a_k$ for any positive integer k. Here,

$$a_1 = 9 \quad \text{and} \quad d = \underline{\qquad} - \underline{\qquad} = -5.$$

Thus, the n-th term is

$$a_n = 9 - 5(n-1) = \underline{\qquad\qquad}.$$

Example 8.2.2. The 11-th term of an arithmetic sequence is 32, and the 19-th term is 72. Find the 100-th term.

Solution. Let a_n be the n-th term of the arithmetic sequence. Then, we have

$$a_{11} = a + 10d = 32$$

$$a_{19} = a + 18d = 72.$$

Solving the system of equations for a and d, we get

$$d = \underline{\qquad} \qquad \text{and} \qquad a = -18.$$

Therefore, the n-th term of the arithmetic sequence is

$$a_n = -18 + (n-1) \cdot 5 = \underline{\qquad\qquad}.$$

Thus, the 100-th term is

$$a_{100} = 5 \cdot 100 - 23 = \underline{\qquad}.$$

Theorem 8.2.2 (Partial Sums of Arithmetic Sequences)

For the arithmetic sequence a_n, the n-th partial sum is

$$S_n = \sum_{k=1}^{n} a_k = n\left(\frac{a_1 + a_n}{2}\right).$$

The sum of n constant numbers is

$$\sum_{k=1}^{n} c = cn, \text{ where } c \text{ is a constant.}$$

The sum of the first n positive integers is

$$\sum_{k=1}^{n} k = \frac{n(n+1)}{2}.$$

Proof. The formula $\sum_{k=1}^{n} c = cn$ is clear.

Note that

$$a_k + a_{n-k} = a_1 + (k-1)d + a_1 + (n-k-1)d = 2a_1 + (n-2)d = a_1 + a_n,$$

for any $1 \leq k \leq n$. Thus,

$$2S_n = (a_1 + a_n) + (a_2 + a_{n-1}) + \cdots (a_n + a_1) = n(a_1 + a_n).$$

Therefore,

$$S_n = n\left(\frac{a_1 + a_n}{2}\right).$$

In particular, when $d = 1$ and $a_1 = 1$, we have $a_n = n$, and

$$S_n = n\left(\frac{1+n}{2}\right) = \frac{n(n+1)}{2}.$$

□

Remark

The formula for the sum of the first n positive integers can also be derived using induction or geometrically by arranging dots into a triangle (see for example https://artofproblemsolving.com/wiki/index.php/Proofs_without_words).

Example 8.2.3. Find the sum of the first 50 odd numbers.

Solution. The sequence of the first 50 odd numbers is an arithmetic sequence with the first term $a_1 = 1$ and the common difference $d = 2$. Thus, the 50-th term is

$$a_{50} = 1 + (50-1) \cdot 2 = \underline{\qquad}.$$

Therefore, the sum of the first 50 odd numbers is

$$S_{50} = 50\left(\frac{1 + \underline{\qquad}}{2}\right) = \underline{\qquad}.$$

Example 8.2.4. Find the following partial sum of an arithmetic sequence:

$$3 + 7 + 11 + 15 + \cdots + 159.$$

Solution. The sequence 3, 7, 11, 15, $\cdots$ is an arithmetic sequence with the first term $a_1 = 3$ and the common difference $d = \underline{\qquad}$. To find the number of terms, we solve for n in the equation

$$a_n = 3 + (n-1) \cdot 4 = 159.$$

Thus, we have

$$\begin{aligned} (n-1) \cdot 4 &= 156 \\ n - 1 &= 39 \\ n &= 40. \end{aligned}$$

Therefore, there are 40 terms in the sequence. Thus, the sum of the sequence is

$$S_{40} = 40\left(\frac{3+159}{2}\right) = \underline{\qquad}.$$

Example 8.2.5. How many terms of the arithmetic sequence $5, 7, 9, \cdots$ must be added to get 572?

Solution. Because the sequence is arithmetic with $a_1 = 5$ and $d = $ ______, we have

$$a_n = 5 + 2(n-1) = \_\_\_\_\_\_.$$

The sum of the first n terms is

$$S_n = n\left(\frac{a_1 + a_n}{2}\right) = n\left(\frac{5 + (2n+3)}{2}\right) = \_\_\_\_\_\_ .$$

To find how many terms must be added to get 572, we solve for n in the equation

$$n(n+4) = 572.$$

Thus, we have

$$\begin{aligned} n^2 + 4n - 572 &= 0 \\ (n+26)(n - \_\_\_\_\_\_) &= 0 \\ n &= \_\_\_\_\_\_ \quad \text{because} \quad n > 0. \end{aligned}$$

Therefore, 22 terms of the arithmetic sequence must be added to get 572.

Exercises

Exercise 8.2.1. Determine whether the sequence is an arithmetic sequence or not and find the n-th term of the sequence.

1) $1-\sqrt{2}$, $1-2\sqrt{2}$, $1-3\sqrt{2}$, $1-4\sqrt{2}$, $\cdots$

2) $\sqrt{3}$, 3, $3\sqrt{3}$, 9, $\cdots$

3) 1, $-\frac{3}{2}$, 2, $-\frac{5}{2}$, 3, $\cdots$

Answer: 1) Yes, $a_n = 1 - n\sqrt{2}$. 2) No. 3) No.

Exercise 8.2.2. Find the partial sum of an arithmetic sequence.

$$\frac{1}{3} + \frac{2}{3} + 1 + \frac{4}{3} + \frac{5}{3} + \cdots + 33$$

Answer: $S_{50} = 1650$.

Exercise 8.2.3. How many terms of the arithmetic sequence 3, 7, 11, $\cdots$ must be added to get 170?

Answer: 9 terms.

8.3 Geometric Sequences

Definition 8.3.1 (Geometric Sequence)

A **geometric sequence** is an exponential function whose n-th term is

$$a_n = a_m r^{n-m},$$

where r is the common ratio of the sequence.

If we denote the first term a_1 as a, then the n-th term of a geometric sequence is given by

$$a_n = ar^{n-1}.$$

♣

Example 8.3.1. Find a_n for the geometric sequence.

1) $2, -10, 50, -250, 1250, \cdots$.

2) $1, \frac{1}{3}, \frac{1}{9}, \frac{1}{27}, \frac{1}{81}, \cdots$.

Solution. Because the sequence is geometric, its n-th term can be written as $a_n = a_1 r^{n-1}$, where $r = \frac{a_{k+1}}{a_k}$ for any positive integer k.

1) Here,

$$a_1 = 2 \quad \text{and} \quad r = \frac{\underline{\qquad}}{\underline{\qquad}} = -5.$$

Thus, the n-th term is

$$a_n = 2(-5)^{n-1} = \underline{\qquad\qquad}.$$

2) Here,

$$a_1 = 1 \quad \text{and} \quad r = \frac{\underline{\qquad}}{\underline{\qquad}} = \frac{1}{3}.$$

Thus, the n-th term is

$$a_n = 1\left(\frac{1}{3}\right)^{n-1} = \underline{\qquad\qquad}.$$

Example 8.3.2. The third term of a geometric sequence is $\frac{63}{4}$, and the sixth term is $\frac{1701}{32}$. Find the fifth term.

Solution. Let a_n be the n-th term of the geometric sequence. Then, we have

$$a_3 = ar^2 = \frac{63}{4} \qquad \text{and} \qquad a_6 = ar^5 = \frac{1701}{32}.$$

Dividing the second equation by the first, we get

$$r^3 = \frac{\frac{1701}{32}}{\frac{63}{4}}$$

$$r = \sqrt[3]{\underline{\qquad}} = \underline{\qquad}.$$

Note that $a_6 = a_5 r$. Thus, the fifth term is

$$a_5 = a_6 \cdot \frac{1}{r} = \frac{1701}{32} \cdot \underline{\qquad} = \frac{589}{16}.$$

Theorem 8.3.2 (Partial Sums of Geometric Sequences)

Given a geometric sequence whose n-th term is $a_n = ar^{n-1}$, the n-th partial sum is

$$S_n = \sum_{k=1}^{n} ar^{k-1} = \frac{a(1-r^n)}{1-r}.$$

Proof. Consider the product

$$S_n(1-r) = S_n - rS_n.$$

We have

$$S_n = a + ar + ar^2 + \cdots + ar^{n-1},$$

and

$$rS_n = ar + ar^2 + ar^3 + \cdots + ar^n.$$

Thus, we get

$$S_n - rS_n = a - ar^n = a(1-r^n).$$

Therefore,

$$S_n = \frac{a(1-r^n)}{1-r}.$$

□

Relation between Arithmetic and Geometric Sequences

The logarithm of a geometric sequence forms an arithmetic sequence. If $\{a_n\}$ is geometric with $a_n = ar^{n-1}$, then $\{\log(a_n)\}$ is arithmetic with

$$\log(a_n) = \log(a) + (n-1)\log(r).$$

Conversely, the exponential of an arithmetic sequence forms a geometric sequence. If $\{b_n\}$ is arithmetic with $b_n = d(n-1) + b$, then $\{r^{b_n}\}$ is geometric with

$$r^{b_n} = r^b \cdot \left(r^d\right)^{n-1}.$$

Example 8.3.3. Find the following partial sum of a geometric sequence:

$$1 + 4 + 16 + \cdots + 4096.$$

Solution. The sequence $1, 4, 16, \cdots$ is a geometric sequence with the first term $a_1 = 1$ and the common ratio $r =$ ______. To find the number of terms, we solve for n in the equation

$$a_n = 1 \cdot 4^{n-1} = 4096$$
$$4^{n-1} = 4096$$
$$n - 1 = \frac{\ln(4096)}{\ln(4)} = \_\_\_\_\_\_$$
$$n = \_\_\_\_\_\_$$

Thus, the sum of the sequence is

$$S_7 = \frac{1(1-4^7)}{1-4} = \_\_\_\_\_\_.$$

Example 8.3.4. Find the sum

$$\sum_{k=1}^{7}\left(-\frac{2}{3}\right)^{k-1}.$$

Solution. Note that when

$$a_1=\left(-\frac{2}{3}\right)^{0}=1 \qquad \text{and} \qquad r=-\frac{2}{3}.$$

Therefore, using the formula for the partial sum of a geometric sequence, we have

$$S_7=\frac{1\left(1-\left(-\frac{2}{3}\right)^7\right)}{1-\left(-\frac{2}{3}\right)}=\underline{\qquad\qquad}.$$

Example 8.3.5. Find the sum

$$\sum_{k=1}^{5}\left(-\frac{5}{3}\right)^{k}.$$

Solution. Note that when

$$a_1=\left(-\frac{5}{3}\right)^{1}=-\frac{5}{3} \qquad \text{and} \qquad r=-\frac{5}{3}.$$

Therefore, using the formula for the partial sum of a geometric sequence, we have

$$S_5=\frac{-\frac{5}{3}\left(1-\left(-\frac{5}{3}\right)^5\right)}{1-\left(-\frac{5}{3}\right)}=\underline{\qquad\qquad}.$$

Definition 8.3.3 (Infinite Series)

An expression of the form

$$\sum_{k=1}^{\infty}a_k=a_1+a_2+a_3+a_4+\cdots$$

is called an **infinite series**.

Definition 8.3.4 (Convergence and Divergence of Geometric Sequences)

An infinite series $\sum\limits_{k=1}^{\infty}a_k$ is said to be **convergent** if the sequence of partial sums $S_n=\sum\limits_{k=1}^{n}a_k$ converges to a finite number. Otherwise, the series is said to be **divergent**.

Given a geometric series

$$\sum_{k=1}^{\infty}ar^{k-1}=a+ar+ar^2+ar^3+\cdots.$$

1) If $|r|<1$, then the series converges to $S=\dfrac{a}{1-r}$.

2) If $|r|\geq 1$, the series diverges.

Example 8.3.6. Determine whether the infinite geometric series is convergent or divergent. If it is convergent, find its sum.

1) $1 - \frac{2}{5} + \frac{4}{25} - \frac{8}{125} + \cdots$ 2) $\sum_{k=1}^{\infty} \frac{1}{4} \cdot \left(\frac{3}{2}\right)^k$ 3) $\sum_{k=1}^{\infty} pq^{k-1}$, $|q| < 1$

Solution. To determine whether the infinite geometric series is convergent or divergent, we identify the first term a and the common ratio r of each series.

1) Here,

$$a = 1 \qquad \text{and} \qquad r = -\frac{2}{5}.$$

Since $|r| < 1$, the series converges to

$$S = \frac{1}{1 - \left(-\frac{2}{5}\right)} = \underline{\qquad}.$$

2) Here,

$$a = \frac{1}{4} \qquad \text{and} \qquad r = \frac{3}{2}.$$

Since $|r| < 1$, the series diverges.

3) Here,

$$a = p \qquad \text{and} \qquad r = q.$$

Since $|q| < 1$, the series converges to

$$S = \frac{p}{1 - q}.$$

Exercises

Exercise 8.3.1. Find the n-th term of the sequence and determine whether the sequence is a geometric sequence, or neither.

1) $\sqrt{2}$, 2, $2\sqrt{2}$, 4, $\cdots$

2) -1, $\frac{4}{3}$, $-\frac{5}{3}$, 2, $\cdots$

Answer: 1) $a_n = \sqrt{2}\left(\sqrt{2}\right)^{n-1} = \sqrt{2}^n$. Yes. 2) $a_n = \frac{(-1)^n(n+2)}{3}$. No.

Exercise 8.3.2. Find the partial sum of a geometric sequence.

1) $\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \cdots + \frac{1}{1024}$

2) $\sum_{k=1}^{n} a \cdot (-b)^{k-1}$

Answer: 1) $S_{10} = \frac{1023}{1024}$. 2) $S_n = a\left(\frac{1-(-b)^n}{1+b}\right)$.

Exercise 8.3.3. Determine whether the infinite geometric series is convergent or divergent. If it is convergent, find its sum.

1) $1 - \frac{5}{2} + \frac{25}{4} - \frac{125}{8} + \cdots$

2) $\sum_{k=1}^{\infty} 3 \cdot \left(-\frac{1}{2}\right)^k$

3) $\sum_{k=1}^{\infty} \frac{1}{(x^2+2)^{k-1}}$

Answer: 1) Divergent. 2) Convergent, $S = 2$. 3) Convergent, $S = \frac{x^2+2}{x^2+1}$.

8.4 The Binomial Theorem

Theorem 8.4.1 (Binomial Theorem)

The **binomial theorem** states that for any positive integer n,

$$(a+b)^n = \sum_{k=0}^{n} \binom{n}{k} a^{n-k} b^k,$$

where $\binom{n}{k}$ are **binomial coefficients** and defined as

$$\binom{n}{k} = \frac{n!}{k!(n-k)!} = \frac{n \cdot (n-1) \cdot \dots \cdot (n-k+1)}{k \cdot (k-1) \cdot \dots \cdot 2 \cdot 1},$$

where $n! = n \cdot (n-1) \cdot \dots \cdot 2 \cdot 1$ is **the factorial** of n. In particular, $\sum_{k=0}^{n} \binom{n}{k} = 2^n$. ♡

Proof. To get the term $a^{n-k}b^k$ in the expansion of $(a+b)^n$ corresponds to choosing k factors of b from n factors of $(a+b)$. The number of ways to choose k factors of b from n factors can be counted first with an order, there are $n(n-1)\cdots(n-k+1)$ ways, then dividing by the number of ways to arrange k factors of b, which is $k(k-1)\cdots 2 \cdot 1$. Thus, the coefficient of $a^{n-k}b^k$ is $\binom{n}{k}$. □

Properties of Binomial Coefficients

The binomial coefficients have the following special values:

$$\binom{n}{0} = \binom{n}{n} = 1, \qquad \binom{n}{1} = \binom{n}{n-1} = n, \qquad \binom{n}{2} = \binom{n}{n-2} = \frac{n(n-1)}{2}.$$

They also satisfy the following relations:

$$\binom{n}{r} = \binom{n}{n-r} = \binom{n-1}{r-1} + \binom{n-1}{r},$$

which can be derived from the definition of binomial coefficients or by comparing coefficients in the expansion of

$$\sum_{k=0}^{n} \binom{n}{k} a^{n-k} b^k = (a+b)^n = (a+b)(a+b)^{n-1} = (a+b) \sum_{k=0}^{n-1} \binom{(n-1)}{k} a^{n-1-k} b^k.$$

The relations can be visualized using Pascal's triangle, where the number in each position is the sum of the two numbers directly above it.

Pascal's Triangle with 8 rows — The number in the n row, k-th column is $\binom{n}{k}$

$$\begin{array}{c}
1 \\
1 \quad 1 \\
1 \quad 2 \quad 1 \\
1 \quad 3 \quad 3 \quad 1 \\
1 \quad 4 \quad 6 \quad 4 \quad 1 \\
1 \quad 5 \quad 10 \quad 10 \quad 5 \quad 1 \\
1 \quad 6 \quad 15 \quad 20 \quad 15 \quad 6 \quad 1 \\
1 \quad 7 \quad 21 \quad 35 \quad 35 \quad 21 \quad 7 \quad 1
\end{array}$$

Example 8.4.1. Calculate the binomial coefficients.

1) $\binom{7}{3}$ 2) $\binom{50}{4}$ 3) $\binom{100}{97}$

Solution. To calculate the binomial coefficients, we use the definition of binomial coefficients. When k is greater than $\frac{n}{2}$, we use the relation $\binom{n}{k} = \binom{n}{n-k}$ to simplify the calculation.

1) We have

$$\binom{7}{3} = \frac{7!}{3!(7-3)!} = \frac{7 \cdot 6 \cdot 5}{3 \cdot 2 \cdot 1} = \underline{\qquad}.$$

2) We have

$$\binom{50}{4} = \frac{50!}{4!(50-4)!} = \frac{50 \cdot 49 \cdot 48 \cdot 47}{4 \cdot 3 \cdot 2 \cdot 1} = \underline{\qquad}.$$

3) We have

$$\binom{100}{97} = \binom{100}{3} = \frac{100!}{3!(100-3)!} = \frac{100 \cdot 99 \cdot 98}{3 \cdot 2 \cdot 1} = \underline{\qquad}.$$

Example 8.4.2. Use the binomial theorem to expand $(x+y)^5$.

Solution. To expand $(x+y)^5$, we use the binomial theorem:

$$(x+y)^5 = \sum_{k=0}^{5} \binom{5}{k} x^{5-k} y^k.$$

As $n = 5$ is not large, using the definition and properties of binomial coefficients directly or the Pascal's triangle, we have

$$\binom{5}{0} = 1, \quad \binom{5}{1} = 5, \quad \binom{5}{2} = \underline{\qquad}, \quad \binom{5}{3} = \underline{\qquad}, \quad \binom{5}{4} = \underline{\qquad}, \quad \binom{5}{5} = \underline{\qquad}.$$

Therefore, we get

$$(x+y)^5 = x^5 + 5x^4y + 10x^3y^2 + \underline{\qquad\qquad\qquad}.$$

Example 8.4.3. Use the binomial theorem to expand $(\sqrt{x}-1)^4$.

Solution. When $n = 4$, the binomial coefficients are

$$\binom{4}{0} = 1, \quad \binom{4}{1} = 4, \quad \binom{4}{2} = \underline{\qquad}, \quad \binom{4}{3} = 4, \quad \binom{4}{4} = 1.$$

Therefore, using the binomial theorem, we have

$$\begin{aligned}(\sqrt{x}-1)^4 &= \sum_{k=0}^{4} \binom{4}{k} (\sqrt{x})^{4-k} (-1)^k \\ &= \sqrt{x}^4 - 4\sqrt{x}^3 + \underline{\qquad\qquad\qquad} \\ &= \underline{\qquad\qquad\qquad\qquad}.\end{aligned}$$

Example 8.4.4. Find the term that contains x^5 in the expansion of $(2x-1)^{10}$.

Solution. To find the term that contains x^5 in the expansion of $(2x-1)^{10}$, we use the binomial theorem:

$$(2x-1)^{10} = \sum_{k=0}^{10} \binom{10}{k} (2x)^{10-k}(-1)^k.$$

The term that contains x^5 corresponds to $10-k =$ ______, or $k=5$. Thus, the term is

$$\binom{10}{5}(2x)^5(-1)^5.$$

Using the properties of binomial coefficients, we have

$$\binom{10}{5} = \frac{10!}{5!(10-5)!} = \frac{10\cdot 9\cdot 8\cdot 7\cdot 6}{5\cdot 4\cdot 3\cdot 2\cdot 1} = \underline{\qquad}.$$

Therefore, the term that contains x^5 is

$$\underline{\qquad\qquad\qquad\qquad}.$$

Example 8.4.5. Find the term that contains x^2 in the expansion of $\left(x^3 - \frac{1}{x}\right)^{12}$.

Solution. To find the term that contains x^2 in the expansion of $\left(x^3 - \frac{1}{x}\right)^{12}$, we use the binomial theorem:

$$\left(x^3 - \frac{1}{x}\right)^{12} = \sum_{k=0}^{12} \binom{12}{k} (x^3)^{12-k}\left(-\frac{1}{x}\right)^k = \sum_{k=0}^{12} (-1)^k x^{\underline{\qquad}}.$$

The term that contains x^2 corresponds to $3(12-k)-k =$ ______, or $k =$ ______. Thus, the x^2 term is

$$\binom{12}{\underline{\qquad}}(-1)^9 x^2 = \underline{\qquad\qquad}.$$

Exercises

Exercise 8.4.1. Evaluate the expression.

1) $\binom{5}{3}$

2) $\binom{5}{3} + \binom{5}{4}$

3) $\sum_{k=0}^{5} \binom{5}{k}$

Answer: 1) 10 2) 15 3) 32

Exercise 8.4.2. Expand the expression.

1) $(2x+y)^6$

2) $\left(x - \frac{1}{x^2}\right)^5$

Answer: 1) $64x^6 + 192x^5y + 240x^4y^2 + 160x^3y^3 + 60x^2y^4 + 12xy^5 + y^6$. 2) $x^5 - 5x^2 + \frac{10}{x} - \frac{10}{x^4} + \frac{5}{x^7} - \frac{1}{x^{10}}$.

Exercise 8.4.3. Find the term containing x^6 in the expansion of $(x+3)^{10}$

Answer: $60480x^6$.

Exercise 8.4.4. Find the term containing no x in the expansion of $\left(4x+\frac{1}{2}x\right)^{10}$.

Answer: 8064.